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Graded Chern-Simons field theory and graded topological D-branes

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Abstract: We discuss graded D-brane systems of the topological A model on a Calabi-Yau threefold (in the absence of worldsheet instanton corrections), by means of their string field theory. We give a detailed analysis of the extended string field action, showing that it satisfies the classical master equation, and construct the associated BV system. The analysis is entirely general and it applies to any collection of D-branes (of distinct grades) wrapping the same special lagrangian cycle, being valid in arbitrary topology. Our discussion employs a $\mathbb{Z}$-graded version of the covariant BV formalism, whose formulation involves the concept of *graded supermanifolds*. We discuss this formalism in detail and explain why $\mathbb{Z}$-graded supermanifolds are necessary for a correct geometric understanding of BV systems. For the particular case of graded D-brane pairs, we also give a direct construction of the master action, finding complete agreement with the abstract formalism. We analyze formation of acyclic composites and show that, under certain topological assumptions, all states resulting from the condensation process of a pair of branes with grades differing by one unit are BRST trivial and thus the composite can be viewed as a closed string vacuum. We prove that there are *six* types of pairs which must be viewed as generally inequivalent. This contradicts the assumption that 'brane-antibrane' systems exhaust the nontrivial dynamics of topological A-branes with the same geometric support.

Keywords: Topological Field Theories, Chern-Simons Theories, D-branes, BRST Symmetry.

Contents

1. Introduction

The issue of D-brane composite formation plays a central role in the study of Calabi-Yau compactifications of open superstrings. It has gradually become clear [1]–[8] that a proper understanding of D-brane condensation processes holds the key to disentangling the relevance of the derived category of coherent sheaves and its A-model analogue, the derived category of Fukaya's category [9, 10, 11] (or rather, a generalization thereof [6]). Since D-brane condensation involves off-shell string dynamics, the most systematic approach to this problem is through the methods of open string field theory. In fact, it seems that 'mirror symmetry with open strings' is best formulated as a (quasi)-equivalence of open string field theories with D-branes, an overarching off-shell version which would subsume all previous statements. A fundamental ingredient in this program is the observation of [1] that the correct description of D-branes in Calabi-Yau compactifications involves an extra datum, an integer-valued quantity called the D-brane's grade. As pointed out in [3], the procedure of including graded D-branes admits a general string field theoretic description (the so-called 'shift completion' of a D-brane category). Taking this fact into account leads to a concrete presentation of the *extended* moduli space of open string vacua, an approach which affords computation of various physical quantities away from points of the geometric moduli space. Moreover, the papers [3, 5] and [6] gave an explicit description of certain subsectors of the relevant string field theories for the case of Calabi-Yau threefolds.

The purpose of this paper is to continue the analysis (initiated in [6]) of a string field-theoretic model relevant for the dynamics of A-type [12] graded *topological* D-branes of a Calabi-Yau threefold compactification *in the absence of worldsheet instanton corrections.* In [6], a sector of the string field theory of such objects was described in terms of a $\mathbb{Z}$-graded version of Chern-Simons field theory living on a special lagrangian cycle. The model captures the off-shell dynamics of topological branes of arbitrary grades wrapped over the cycle. As sketched in that paper, the classical moduli space of vacua of this theory allows one to recover a piece of the extended boundary moduli space, an object which plays a crucial role in the homological mirror symmetry program [13]. More precisely, points of the extended moduli space can be described as *generalized D-brane composites*, obtained by condensing boundary condition changing states between *graded* D-branes. This suggest that one can extract physical information about such points by studying the quantum dynamics of the resulting theory.

Since the theory written down in [6] involves higher rank forms, its quantization must deal with the issue of reducible gauge algebras. Therefore, a correct analysis of this theory requires the full force of the Batalin-Vilkovisky formalism. The purpose of the present paper is to carry out the classical part of this analysis, thus providing a precise starting point for a study of quantum dynamics. Applying such methods, we will be able to recover the extended action already written down in [6], which plays the role of (classical) master action for our theory. This enables us to show that the model constructed in [6] is a consistent starting point for a quantum-mechanical analysis. As an application, we give a detailed discussion of graded D-brane pairs, thus obtaining a realization of ideas proposed in [14], though in a somewhat different context.

Our investigation reveals that there are six types of D-brane pairs which are physically inequivalent in general. This confirms the $\mathbb{Z}_6$ periodicity of the D-brane grade suggested in [1] from worldsheet considerations and contradicts the assumption that 'brane-antibrane' systems exhaust the nontrivial dynamics of topological A-branes with the same geometric support.

The paper is organized as follows. We start in section 2 by recalling the construction of the classical string field action of [6]. In section 3, we discuss the associated extended string field theory, giving a detailed presentation of the structures involved. Though this section is somewhat technical, a clear description of these structures is crucial for a correct understanding of latter work. The central objects are the so-called extended boundary product and extended bilinear form introduced in [6], whose construction we explain in detail. In section 4, we proceed to show that the extended action discussed in section 3 satisfies the classical master equation with respect to the antibracket induced by a certain odd symplectic form (which coincides with the extended bilinear form up to sign factors and a shift of grading). This establishes the fact that the the extended action plays the role of classical BV action for our systems. The proof, which is completely general, makes use of the geometric version of the BV formalism, as discussed, for example, in [15]. Our approach is in fact a certain modification of usual covariant framework, which differs from the latter by incorporating the ghost grading. This modified formalism, which is necessary for a correct description of BV systems, involves the concept of *graded supermanifolds* which was recently introduced in [16]. We therefore start with a general exposition of the geometric $\mathbb{Z}$-graded formalism, which is of independent interest for foundational studies of BV quantization. We then apply this framework to the extended string field theory of section 3. This allows us to give a complete construction of the relevant BV system, and a very concise and general proof of the fact that the extended action satisfies the classical master equation. We also identify the classical gauge which leads to the unextended string field theory. After discussing the form of the BRST operator in this gauge, we proceed with a discussion of the particular case of graded D-brane pairs. Section 5 recalls the open string interpretation of the various data discussed in section 2, and gives their concrete description for such systems. In section 6, we consider composite formation for D-brane pairs and the issue of acyclic condensates. After discussing the worldsheet BRST cohomology and explaining how it distinguishes between the various pairs, we explain the interpretation of string field vacua as points of the extended moduli space, and give an explicit construction of acyclic composites for certain graded D-brane pairs whose relative grading equals one. This gives a concrete realization of suggestions made in [14].

While the geometric BV framework of section 4 is extremely powerful and general, it does not explain the origin of the various components of the extended string field. To gain some insight into this issue, we proceed in section 7 with a direct construction of the BV action for the case of graded pairs. We show that the more familiar component approach leads to an action which can be viewed as the expression of the extended action of section 3 in a particular system of linear coordinates, and show how the various ghosts and antifields arise in standard manner by performing the BV-BRST resolution of our (closed, but generally reducible) gauge algebras. We also give a discussion of the relation between pairs with arbitrarily high relative grade. Section 8 connects these results with the geometric approach of section 4, and in particular gives a very concise formulation of the BV-BRST algorithm. This synthetic description should be useful for understanding the structure of the gauge algebra of systems containing more than two graded branes. We end in section 9 by presenting our conclusions and a few directions for further research.

2. A string field theory for graded topological D-branes

This section describes the string field theory of a collection of graded topological D-branes wrapping the same special lagrangian cycle of a Calabi-Yau threefold. This theory was written down in [6] starting from a worldsheet analysis and using the framework developed in [2, 3].

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2.1 Graded D-branes

We start by recalling some basic concepts introduced in [1] (see also [4] and [6]). We are interested in so-called *graded topological D-branes* of the A-model compactified on a Calabi-Yau threefold X. Recall from [17] that an *ungraded* topological D-brane can be described as a pair (L, E) where L is a (connected) special lagrangian cycle of X and E is a flat vector bundle on L.[1] The generalization to *graded* D-branes [1, 6] is obtained by replacing L with a graded version [18], which for practical purposes amounts to fixing an integer n (the brane's *grade*) and a certain orientation of L which depends on n. As discussed in [1, 4, 6], the worldsheet U(1) charge in the boundary condition changing sectors between two graded D-branes wrapping L is then shifted by $\pm n$, while the boundary products in various sectors contain extra signs. Moreover, the boundary metric receives grade-dependent signs in boundary condition changing sectors, due to the fact that the relevant orientation of L depends on the branes' grades.

2.2 Graded Chern-Simons field theory as a string field theory for graded D-branes

As explained in detail in [6], it is possible to describe certain graded D-brane systems through a generalization of Chern-Simons field theory. To be specific, we consider a collection of graded topological A-type branes wrapping *the same* special lagrangian cycle L in X. The main assumption for what follows is that no two D-branes of this collection have the same grade. This allows one to label the branes by their grades, which form a finite or infinite set of integers. If a_n denotes the D-brane of grade n, then we denote by E_n its underlying bundle. This notation includes the specification of a flat structure (flat connection) on each E_n.

With these hypotheses, it was shown in [6] that the string field theory of the system is a graded version of Chern-Simons field theory, which generalizes both the usual Chern-Simons description of [17] and the supergroup Chern-Simons proposal of [14]. This describes the dynamics of so-called 'degree one graded connections' [19] on the graded (super-)bundle $\mathbf{E} = \oplus_n E_n$. To define the theory, one must first describe the so-called *total boundary space* $\mathcal{H}$ of [6], which consists of the off-shell states of open strings stretching between our branes.

2.2.1 The total boundary algebra

We start by considering the algebra $\mathcal{E}$ of endomorphisms of $\mathbf{E}$. This admits a natural $\mathbb{Z}$-grading induced by the bundle decomposition:

$$\mathrm{End}(\mathbf{E}) = \oplus_k \mathrm{End}_k(\mathbf{E}) \,, \tag{2.1}$$

where:

$$\mathrm{End}_k(\mathbf{E}) = \oplus_{n-m=k} \mathrm{Hom}(E_m, E_n) \,. \tag{2.2}$$

[1] We remind the reader that this is a vector bundle endowed with a flat structure, i.e. the equivalence class of a family of local trivializations whose transition functions are constant. Specifying a flat structure amounts to giving a gauge-equivalence class of flat connections.

If f is a morphism from E_m to E_n, then its degree with respect to this grading is given by:

$$\Delta(f) = n - m\,. \tag{2.3}$$

It is easy to see that the composition of morphisms is homogeneous of degree zero with respect to this grading:

$$\Delta(f \circ g) = \Delta(f) + \Delta(g)\,. \tag{2.4}$$

It follows that $\mathcal{E}$ (with multiplication given by composition of morphisms) forms a graded associative algebra with respect to the grading induced by Δ.

The next step is to consider the tensor product $\mathcal{H} = \Omega^*(L) \otimes \mathcal{E}$ between the exterior algebra of L and the endomorphism algebra $\mathcal{E}$. Since both algebras are $\mathbb{Z}$-graded, $\mathcal{H}$ is endowed with gradings rk and Δ induced from its components, as well as with the total grading $|\cdot| = rk + \Delta$. On decomposable elements $u = \rho \otimes f$, these are given by:

$$rku = rk\rho\,, \qquad \Delta(u) = \Delta(f)\,, \qquad |u| = rk\rho + \Delta(f)\,. \tag{2.5}$$

An arbitrary element u of $\mathcal{H}$ can be viewed as a matrix $u = (u_{mn})$, where $u_{mn} \in \Omega^*(L) \otimes \Gamma(\mathrm{Hom}(E_m, E_n))$ is a bundle-valued form. Then $\Delta(u_{mn}) = n - m$ and $|u_{mn}| = rku_{mn} + n - m$. The space $\mathcal{H}$ is also endowed with a canonical multiplication $\bullet$ (the 'total boundary product' of [6]), induced from the multiplicative structure of its tensor components. On decomposable elements $u = \rho \otimes f$ and $v = \eta \otimes g$, this is given by:

$$u \bullet v = (-1)^{\Delta(f) rk\eta} (\rho \wedge \eta) \otimes (f \circ g)\,. \tag{2.6}$$

Up to the sign prefactor, the right hand side is simply the usual wedge product of $\mathrm{End}(\mathbf{E})$-valued forms, which includes composition of bundle morphisms on the coefficients:

$$u \wedge v = (\rho \wedge \eta) \otimes (f \circ g)\,. \tag{2.7}$$

This allows us to write (2.6) in a slightly more familiar form:

$$u \bullet v = (-1)^{\Delta(u) rkv} u \wedge v\,. \tag{2.8}$$

The boundary product is homogeneous of degree zero with respect to the total grading:

$$|u \bullet v| = |u| + |v|\,. \tag{2.9}$$

Hence $\mathcal{H}$ becomes a graded associative algebra when endowed with the grading $|\cdot|$ and the product $\bullet$. This well-known construction of a graded associative structure on the tensor product from similar structures on components is usually denoted by:

$$\mathcal{H} = \Omega^*(L) \hat{\otimes} \mathcal{E}\,, \tag{2.10}$$

where the hat above the tensor product indicates that the multiplication and grading on the resulting space are constructed in the canonical manner discussed above.

A supplementary datum is provided by the existence of a natural differential d on the product algebra $\mathcal{H}$. This is the exterior differential on $\mathrm{End}(\mathbf{E})$-valued forms, twisted with the direct sum connection $A = \oplus_n A_n$. Flatness of A_n assures that $d^2 = 0$, and definition (2.6) implies that d acts as a graded derivation of the product $\bullet$ (with respect to the total grading):

$$d(u \bullet v) = (du) \bullet v + (-1)^{|u|} u \bullet (dv) \,. \tag{2.11}$$

Moreover, one has:

$$|du| = |u| + 1 \,. \tag{2.12}$$

The final element needed in the construction is a 'trace' on $\mathcal{H}$. This is induced by the natural traces on $\Omega^*(L)$ and $\mathcal{E}$, which are defined as follows. For complex-valued forms $\rho \in \Omega^*(L)$, we define:

$$\mathrm{Tr}_\Omega(\rho) = \int_L \rho \,, \tag{2.13}$$

while for End($\mathbf{E}$)-valued morphisms f we have the supertrace of [20]:

$$\mathrm{str}(f) = \sum_m (-1)^m \, \mathrm{tr}_m(f_{mm}) \,, \tag{2.14}$$

where tr_m is the fiberwise trace in the bundle End(E_m). Note that str(f) is a complex-valued function defined on L. When all components E_n of $\mathbf{E}$ have grades n of the same parity (which in the language of [20] amounts to taking the even or odd component of $\mathbf{E}$ to be the zero bundle), the supertrace (2.14) reduces to $\pm$ the ordinary fiberwise trace on the bundle End($\mathbf{E}$).

Both traces are graded-symmetric with respect to the natural degrees on their spaces of definition:

$$\mathrm{Tr}_\Omega(\rho \wedge \eta) = (-1)^{rk\rho rk\eta} \, \mathrm{Tr}_\Omega(\eta \wedge \rho) \,, \quad \mathrm{str}(f \circ g) = (-1)^{\Delta(f)\Delta(g)} \, \mathrm{str}(g \circ f) \,. \tag{2.15}$$

Since $\mathrm{Tr}_\Omega(u \wedge v)$ and $\mathrm{str}(f \circ g)$ vanish unless $rk\rho + rk\eta = 3$ (remember that L is a 3-cycle!), respectively $\Delta(f) + \Delta(g) = 0$, the graded symmetry properties are equivalent with:

$$\mathrm{Tr}_\Omega(\rho \wedge \eta) = \mathrm{Tr}_\Omega(\eta \wedge \rho) \quad \text{and} \quad \mathrm{str}(f \circ g) = (-1)^{\Delta(f)} \, \mathrm{str}(g \circ f) \,. \tag{2.16}$$

Using (2.15), we define a trace on $\mathcal{H}$ which on decomposable elements $u = \rho \otimes f$ is given by:

$$\mathrm{Tr}_{\mathcal{H}}(u) = \int_L \mathrm{str}(f)\rho \,. \tag{2.17}$$

It is easy to check that this is graded-symmetric:

$$\mathrm{Tr}_{\mathcal{H}}(u \bullet v) = (-1)^{|u||v|} \, \mathrm{Tr}_{\mathcal{H}}(v \bullet u) \,. \tag{2.18}$$

It immediately follows that the (nondegenerate) bilinear form on $\mathcal{H}$, defined through:

$$\langle u, v \rangle := \mathrm{Tr}_{\mathcal{H}}(u \bullet v) = \int_L \mathrm{str}(u \bullet v) \,, \tag{2.19}$$

is graded-symmetric as well:

$$\langle u, v \rangle = (-1)^{|u||v|} \langle v, u \rangle \,. \tag{2.20}$$

It is clear that $\langle u,v\rangle$ vanishes on bi-homogeneous elements unless both of the conditions $rku + rkv = 3$ and $\Delta(u) + \Delta(v) = 0$ are satisfied. It follows that non-vanishing of $\langle u,v\rangle$ requires $|u| + |v| = 3$, for elements homogeneous with respect to the total degree. Due to this selection rule, the graded symmetry property is in fact equivalent with:

$$\langle u,v\rangle = \langle v,u\rangle\,. \tag{2.21}$$

The last properties we shall need are invariance of the bilinear form with respect to the total boundary product and differential:

$$\langle du,v\rangle + (-1)^{|u|}\langle u,dv\rangle = 0\,, \qquad \langle u \bullet v, w\rangle = \langle u, v\bullet w\rangle\,. \tag{2.22}$$

These follows easily upon using the properties of the differential and supertrace. We end by noting that the trace and bilinear form can be expressed in more familiar language as follows:

$$\mathrm{Tr}_{\mathcal{H}}(u) = \int_L \mathrm{str}(u)\,, \qquad \langle u,v\rangle = \int_L (-1)^{\Delta(u)rkv}\,\mathrm{str}(u\wedge v)\,, \tag{2.23}$$

if one extends the supertrace to bundle-valued forms through:

$$\mathrm{str}(\rho\otimes f) := \mathrm{str}(f)\rho\,. \tag{2.24}$$

2.2.2 The string field action and its gauge algebra

The string field theory of [6] is described by the action:

$$S(\phi) = \int_L \mathrm{str}\left[\frac{1}{2}\phi\bullet d\phi + \frac{1}{3}\phi\bullet\phi\bullet\phi\right]. \tag{2.25}$$

This is defined on the component $\mathcal{H}^1 = \{\phi\in\mathcal{H}||\phi| = 1\}$ of the total boundary space. It can also be written in the form:

$$S(\phi) = \frac{1}{2}\langle\phi, d\phi\rangle + \frac{1}{3}\langle\phi,\phi\bullet\phi\rangle\,, \tag{2.26}$$

which is discussed, for example, in [2]. As explained in more detail below, the physical interpretation of the defining data is as follows. d plays the role of a 'total worldsheet BRST charge' on a certain collection of (topological) open string sectors. The product $\bullet$ is a total open string product for that collection and the bilinear form $\langle\cdot,\cdot\rangle$ is a 'total BPZ form' (total topological metric).

The action (2.25) is invariant with respect to infinitesimal gauge transformations of the form:

$$\phi\to\phi+\delta_\alpha\phi = \phi - d\alpha - [\phi,\alpha]_\bullet\,, \tag{2.27}$$

where $\delta_\alpha\phi = -d\alpha - [\phi,\alpha]_\bullet$, with $\alpha\in\mathcal{H}^0 = \oplus_{\substack{m,n\\ m\geq n}}\Gamma(\mathrm{Hom}(E_m,E_n))\otimes\Omega^{m-n}(L)$ a charge zero element ($|\alpha| = 0$) of $\mathcal{H}$. In these relations, $[\cdot,\cdot]_\bullet$ denotes the graded commutator in the total boundary algebra, which on arbitrary homogeneous elements is given by:

$$[u,v]_\bullet = u\bullet v - (-1)^{|u||v|}v\bullet u\,. \tag{2.28}$$

It is easy to check that:

$$\delta_\alpha \delta\beta\phi - \delta_\beta\delta_\alpha\phi = \delta_{[\alpha,\beta]_\bullet}\phi\,, \tag{2.29}$$

so that the Lie algebra of transformations of the form (2.27) closes off-shell. Note that $[\alpha,\beta]_\bullet = \alpha\bullet\beta - \beta\bullet\alpha$ since α and β have vanishing U(1) charge. In fact, relation (2.28) shows that our gauge transformations give a representation of the (infinite-dimensional) Lie algebra $\mathbf{g} = (\mathcal{H}^0, [\cdot,\cdot]_\bullet)$, which is a subalgebra of the graded Lie algebra $(\mathcal{H}, [\cdot,\cdot]_\bullet)$. The gauge group $\mathcal{G}$ results by exponentiation of $\mathbf{g}$. This infinite-dimensional group is rather exotic, since its generators are higher rank forms on the cycle L. Note that we insist on considering *all* elements of $\mathcal{H}^0$ as generators of this group, even though we start with a particular background flat connection on $E = \oplus_n E_n$ which happens to split as a direct sum $\oplus_m A_m$. Even for a direct sum background, one *cannot* restrict to the 'diagonal' subalgebra $\oplus_m\Gamma(\mathrm{Hom}(E_m,E_m))(L)$ of $\mathbf{g}$. Inclusion of non-diagonal generators is necessary for consistency of the string field interpretation, since such a decomposition of the connection is accidental, and one can deform away from direct sum backgrounds by condensing boundary condition changing states [6].

2.3 The open string interpretation

The precise interpretation of (2.25) in terms of open A-type strings arises upon applying the general formalism discussed in [2, 3]. For this, one considers the decomposition:

$$\mathcal{H} = \oplus_{m,n}\mathcal{H}_{nm}\,, \tag{2.30}$$

where $\mathcal{H}_{nm} = \Omega^*(L)\otimes\Gamma(\mathrm{Hom}(E_m,E_n))$ and notices that the product $\bullet$ and differential d are compatible with it in the sense that $\bullet$ vanishes on $\mathcal{H}_{kn'}\times\mathcal{H}_{nm}$ if $n\neq n'$ and takes $\mathcal{H}_{kn}\times\mathcal{H}_{nm}$ into $\mathcal{H}_{km}$, while d takes $\mathcal{H}_{nm}$ into $\mathcal{H}_{nm}$. This implies that the collection of spaces $\mathcal{H}_{nm}$ can be viewed as the morphism spaces of a differential graded category built on the objects a_n (which, due to our assumption, are in bijection with the grades n present in the system). The category interpretation results upon defining $\mathrm{Hom}(a_m,a_n) := \mathcal{H}_{nm}$. One can further check that the bilinear form $\langle\cdot,\cdot\rangle$ is compatible with the decomposition (2.30), in the sense that it vanishes on $\mathcal{H}_{m'n'}\times\mathcal{H}_{nm}$ unless $n'=n$ and $m'=m$. Then the general discussion of [2] suggests that $\mathcal{H}_{nm}$ should be interpreted as the (off-shell) state space of open topological strings stretching from a_m to a_n. This interpretation is indeed valid, and can be recovered from the topological A-model as discussed in [6].

For example, the fact that the natural grading on $\mathcal{H}_{nm}$ is given by $|\cdot|$ implies that the worldsheet U(1) charge of states for the string stretching from a_m to a_n is given by:

$$|u| = rku + n - m\,, \tag{2.31}$$

which agrees with the observation of [1] that the U(1) charge is shifted in boundary condition changing sectors connecting two graded topological D-branes. A direct construction of the string field theory (2.25) can be found in the paper [6], which takes the sigma model perspective as a starting point.

3. The extended string field theory

The theory (2.25) can be extended in a manner reminiscent of that discussed in [21]. This extension was already written down in [6], which gave a very short discussion of its structure. Here we give a more complete exposition. As we shall show later, the extended action obtained in this manner coincides with the tree-level BV action of the theory (2.25); this parallels what happens for the ungraded bosonic open string field theories discussed in [21].

Since we start with an action based on a graded super-bundle, the various objects involved in the extension procedure are somewhat subtle and we shall give a careful discussion of their construction. We warn the reader that a cursory reading of the present section may lead to serious misunderstanding of our sign conventions.

3.1 The extended boundary data

In order to formulate the extended theory, we must define an *extended boundary algebra* $(\mathcal{H}_e, d, *)$, a differential graded associative algebra which extends $(\mathcal{H}, d, \bullet)$. We also need an *extended topological metric* $\langle\cdot,\cdot\rangle_e$, a graded-symmetric nondegenerate bilinear form on $\mathcal{H}_e$ which extends the BPZ form $\langle\cdot,\cdot\rangle$. We shall consider the three elements $*$, d and $\langle\cdot,\cdot\rangle_e$ in turn.

The extended boundary algebra $(\mathcal{H}_e, *)$ is obtained (as in [21]) by considering a (complex) Grassmann algebra G and constructing the graded associative algebra $\mathcal{H}_e = \Omega^*(L)\hat{\otimes}\mathcal{E}\hat{\otimes}G$. The extended boundary product $*$ will be the standard product on this algebra, to be discussed below. The Grassmann algebra G[2] (whose elements we denote by $\alpha, \beta \ldots$) comes endowed with the Grassmann degree g and a multiplication which we write as juxtaposition. We allow G to have any number of odd generators, which we denote by ξ^μ. An element of G has the form:

$$\alpha = \alpha_0 + \sum_{k\geq 1}\sum_{\mu_1<\ldots<\mu_k} \alpha_{\mu_1\ldots\mu_k}\xi^{\mu_1}\cdots\xi^{\mu_k}\,, \tag{3.1}$$

where α_0 and $\alpha_{\mu_1\ldots\mu_k}$ are complex numbers. We note the existence of an evaluation map ev_G from G to $\mathcal{C}$, which projects out the odd generators:

$$ev_G(\alpha) = \alpha_0\,. \tag{3.2}$$

Since we shall tensor with G, we will encounter various $\mathbb{Z}_2$-valued degrees, for which we use the following convention. We let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\hat{0}, \hat{1}\}$, where $\hat{1}$ is the unit. For an element $t \in \mathbb{Z}_2$, we define the power $(-1)^t$ by picking any representative for t in the covering space $\mathbb{Z}$; this is clearly well-defined since $(-1)^2 = +1$. If n is an integer and t is an element of $\mathbb{Z}_2$, then $nt \in \mathbb{Z}_2$ is the product of t with the mod 2 reduction of n.

Associativity of $\hat{\otimes}$ allows us to view $\mathcal{H}_e$ as either of the tensor products $\mathcal{H}\hat{\otimes}G$ or $\Omega^*(L)\hat{\otimes}\hat{\mathcal{E}}$, where $\hat{\mathcal{E}} := \mathcal{E}\hat{\otimes}G$. It will be useful to discuss the multiplicative structure of $\mathcal{H}_e$ from both of these perspectives. For this, we first recall the multiplicative structure on the factor $\hat{\mathcal{E}}$ of the second presentation.

[2] All of the constructions of this paper can in fact be carried out with an *arbitrary* commutative superalgebra G with unit.

3.1.1 The algebra $\hat{\mathcal{E}}$ of Grassmann-valued sections of $\mathrm{End}(\mathbf{E})$

The space $\hat{\mathcal{E}}$ is the graded associative algebra of Grassmann-valued sections of $\mathrm{End}(\mathbf{E})$. If $\hat{f} = f \otimes \alpha$ and $\hat{g} = g \otimes \beta$ are two decomposable elements of $\hat{\mathcal{E}}$ (with f, g elements of $\mathcal{E}$ and α, β elements of G), then their canonical multiplication is given by:

$$\hat{f}\hat{g} = (-1)^{g(\alpha)\Delta(g)}(f \circ g) \otimes (\alpha\beta)\,. \tag{3.3}$$

One can extend the composition of morphisms $\circ$ from $\mathcal{E}$ to a naive multiplication on $\hat{\mathcal{E}}$ given by:

$$\hat{f} \circ \hat{g} = (f \circ g) \otimes (\alpha\beta)\,. \tag{3.4}$$

In terms of this naive product, the defining relation (3.3) becomes:

$$\hat{f}\hat{g} = (-1)^{g(\hat{f})\Delta(\hat{g})}\hat{f} \circ \hat{g}\,, \tag{3.5}$$

where $g(\hat{f}) = g(\alpha)$ and $\Delta(\hat{g}) = \Delta(g)$ are the degrees induced on $\hat{\mathcal{E}}$ by the $\mathbb{Z}_2$-grading of G and the $\mathbb{Z}$-grading of $\mathcal{E}$. Note that $g(\hat{f})$ is simply the grassmannality of $\hat{f}$, while $\Delta(\hat{g}) = n - m$ if $\hat{g}$ is a morphism from E_m to E_n.

The algebra $\hat{\mathcal{E}}$ is endowed with the total $\mathbb{Z}_2$-valued grading induced by the sum $\sigma = \Delta \pmod 2 + g$ between the mod 2 reduction of Δ and the Grassmann degree g on G:

$$\sigma(f \otimes \alpha) = \Delta(f) \pmod 2 + g(\alpha)\,. \tag{3.6}$$

The product (3.3) is homogeneous of degree zero with respect to this grading:

$$\sigma(\hat{f}\hat{g}) = \sigma(\hat{f}) + \sigma(\hat{g})\,. \tag{3.7}$$

The supertrace *str* on $\mathcal{E}$ extends to a functional str_e on $\hat{\mathcal{E}}$, which associates a *Grassmann-valued* function $\mathrm{str}_e(\hat{f})$ (defined on L) to each Grassmann-valued section of $\hat{f}$ of $\mathrm{End}(\mathbf{E})$. On decomposable elements $\hat{f} = f \otimes \alpha$, this 'extended supertrace' is given by:

$$\mathrm{str}_e(f \otimes \alpha) := \mathrm{str}(f) \otimes \alpha\,. \tag{3.8}$$

It is easy to check that the extended supertrace has the property:

$$\mathrm{str}_e(\hat{f}\hat{g}) = (-1)^{\sigma(\hat{f})\sigma(\hat{g})}\,\mathrm{str}_e(\hat{g}\hat{f})\,. \tag{3.9}$$

3.1.2 The multiplicative structure on $\mathcal{H}_e$

A decomposable element $\hat{u}$ of $\mathcal{H}_e$ can be presented as:

$$\hat{u} = \rho \otimes f \otimes \alpha = u \otimes \alpha = \rho \otimes \hat{f}\,, \tag{3.10}$$

where ρ is a (complex-valued) form on L, f is an endomorphism of $\mathbf{E}$ and α is an element of the Grassmann algebra G. In this relation, we defined $u = \rho \otimes f$ (an element of $\mathcal{H} = \Omega^*(L) \otimes \mathcal{E}$) and $\hat{f} = f \otimes \alpha$ (an element of $\hat{\mathcal{E}} = \mathcal{E} \otimes G$). If $\hat{v} = \eta \otimes g \otimes \beta = v \otimes \beta = \eta \otimes \hat{g}$

is another element of $\mathcal{H}_e$ (with $v = \eta \otimes g$ and $\hat{g} = g \otimes \beta$), then the canonical product $*$ in $\mathcal{H}_e$ is given by:[3]

$$\begin{aligned} \hat{u} * \hat{v} &= (-1)^{g(\alpha)|v|}(u \bullet v) \otimes (\alpha\beta) \\ &= (-1)^{(g(\alpha)+\Delta(f))rk\eta}(\rho \wedge \eta) \otimes (\hat{f}\hat{g}) \\ &= (-1)^{g(\alpha)|v|+\Delta(f)rk\eta}(\rho \wedge \eta) \otimes (f \circ g) \otimes (\alpha\beta)\,. \end{aligned} \tag{3.11}$$

The first two equations correspond to viewing $\mathcal{H}_e$ as $\mathcal{H}\hat{\otimes}G$ and $\Omega^*(L)\hat{\otimes}\hat{\mathcal{E}}$ respectively. The last treats $\mathcal{H}_e$ as the triple tensor product $\Omega^*(L)\hat{\otimes}\mathcal{E}\hat{\otimes}G$.

The extended boundary space is equipped with the total ($\mathbb{Z}_2$-valued) degree deg induced from its components:

$$\deg(\hat{u}) = (rk\rho + \Delta(f)) \pmod 2 + g(\alpha) = |u| \pmod 2 + g(\alpha) = rk\rho \pmod 2 + \sigma(\hat{f})\,, \tag{3.12}$$

on elements $\hat{u}$ of the form (3.10). The extended boundary product (3.11) is homogeneous of degree zero with respect to this grading:

$$\deg(\hat{u} * \hat{v}) = \deg(\hat{u}) + \deg(\hat{v})\,. \tag{3.13}$$

3.1.3 The trace and bilinear form on $\mathcal{H}_e$

The extended boundary space is endowed with a trace Tr_e which on decomposable elements $\hat{u} = \rho \otimes f \otimes \alpha$ takes the form:

$$\mathrm{Tr}_e(\hat{u}) = \int_L \rho\, \mathrm{str}(f)\alpha = \int_L \rho\, \mathrm{str}_e(\hat{f}) = \mathrm{Tr}_{\mathcal{H}}(u) \otimes \alpha\,. \tag{3.14}$$

This associates an element of G to every element of $\mathcal{H}_e$. It is easy to check that the extended trace is graded-symmetric with respect to the total degree:

$$\mathrm{Tr}_e(\hat{u} * \hat{g}) = (-1)^{\deg\hat{u}\deg\hat{v}}\,\mathrm{Tr}_e(\hat{v} * \hat{u})\,. \tag{3.15}$$

One can also write:

$$\mathrm{Tr}_e(\hat{u}) = \int_L \mathrm{str}_e(\hat{u})\,, \tag{3.16}$$

upon extending str_e to $\mathcal{H}_e$ through:

$$\mathrm{str}_e(\rho \otimes \hat{f}) = \rho\, \mathrm{str}_e(\hat{f})\,. \tag{3.17}$$

[3] It might be useful to show that the product $*$ is indeed associative. Consider degree one Grassmann valued forms in $\mathcal{H}_e$:

$$\hat{a} * (\hat{b} * \hat{c}) = \hat{a} * (-1)^{(1-rkb)rkc+g(b)\Delta(c)}\hat{b}\hat{c} = (-1)^{(1-rkb)rkc+g(b)\Delta(c)+(1-rka)(rkb+rkc)+g(a)(\Delta(b)+\Delta(c))}\hat{a}\hat{b}\hat{c}\,.$$

On the other hand

$$(\hat{a} * \hat{b}) * \hat{c} = (-1)^{(1-rka)rkb+g(a)\Delta b}(\hat{a}\hat{b}) * \hat{c} = (-1)^{(1-rka)rkb+g(a)\Delta(b)+(2-rka-rkb)rkc+(g(a)+g(b))\Delta(c)}\hat{a}\hat{b}\hat{c}\,.$$

Since the signs are the same in both cases the product $*$ satisfies associativity.

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We next introduce a (nondegenerate) bilinear form on $\mathcal{H}_e$ through:

$$\langle \hat{u}, \hat{v} \rangle_e := \mathrm{Tr}_e(\hat{u} * \hat{v}) = \int_L \mathrm{str}_e(\hat{u} * \hat{v}) \,. \tag{3.18}$$

Property (3.15) assures graded-symmetry of this form with respect to the total degree on the extended boundary space:

$$\langle \hat{u}, \hat{v} \rangle_e = (-1)^{\deg \hat{u} \ \deg \hat{v}} \langle \hat{v}, \hat{u} \rangle_e \,. \tag{3.19}$$

It is also easy to check invariance of the extended bilinear form with respect to the extended boundary product:

$$\langle \hat{u} * \hat{v}, \hat{w} \rangle_e = \langle \hat{u}, \hat{v} * \hat{w} \rangle_e \,. \tag{3.20}$$

We finally note the worldsheet charge selection rule:

$$\langle \hat{u}, \hat{v} \rangle_e = 0 \qquad \text{unless } |\hat{u}| + |\hat{v}| = 3 \,. \tag{3.21}$$

Since the supertrace only couples the component u_{mn} to v_{nm}, one has in fact separate selection rules for the gradings Δ and rk:

$$\langle \hat{u}, \hat{v} \rangle_e = 0 \qquad \text{unless } rk\hat{u} + rk\hat{v} = 3 \qquad \text{and} \qquad \Delta(\hat{u}) + \Delta(\hat{v}) = 0 \,, \tag{3.22}$$

for bi-homogeneous elements u and v.

3.1.4 Expression of the extended product in terms of the wedge product and 'twisted wedge product' of Grassmann-valued forms with coefficients in End(**E**)

It is possible to express the extended boundary data discussed above in somewhat more familiar language as follows. Upon regarding $\mathcal{H}_e$ as the tensor product $\Omega^*(L)\hat{\otimes}\hat{\mathcal{E}}$, one has the usual wedge product:

$$\hat{u} \wedge \hat{v} = (\rho \wedge \eta) \otimes (\hat{f} \circ \hat{g}) \,, \tag{3.23}$$

which uses the composition (3.4) of Grassmann-valued bundle morphisms. One can also define a 'twisted wedge product' by:[4]

$$\hat{u} \boldsymbol{\wedge} \hat{v} = (\rho \wedge \eta) \otimes (\hat{f}\hat{g}) \,. \tag{3.24}$$

The idea behind this definition is that, since $\hat{u}$ and $\hat{v}$ are forms with Grassmann-valued coefficients in a *graded* bundle, it is natural to consider a 'wedge product' which includes multiplication with respect to the natural product (3.3) on the coefficient algebra $\hat{\mathcal{E}}$.

It is easy to see that:

$$\hat{u} \boldsymbol{\wedge} \hat{v} = (-1)^{g(\hat{u})\Delta(\hat{v})} \hat{u} \wedge \hat{v} \tag{3.25}$$

[4]The twisted wedge product $\boldsymbol{\wedge}$ was used in [6], where it was denoted simply by $\wedge$, since it is the natural extension of the wedge product of bundle-valued forms to the graded case. In the present paper, we reserve the notation $\wedge$ (later simply written as juxtaposition) for the usual wedge product built with the ordinary composition of morphisms.

With these definitions, equation (3.11) gives:

$$\hat{u} * \hat{v} = (-1)^{(g(\hat{u})+\Delta(\hat{u}))rk\hat{v}} \hat{u} \boldsymbol{\wedge} \hat{v} = (-1)^{g(\hat{u})|\hat{v}|+\Delta(\hat{u})rk\hat{v}} \hat{u} \wedge \hat{v} \,, \tag{3.26}$$

where we extended the grades Δ, g and rk to $\mathcal{H}_e$ in the obvious manner: $\Delta(\rho \otimes f \otimes \alpha) := \Delta(f)$, $g(\rho \otimes f \otimes \alpha) := g(f)$, $rk(\rho \otimes f \otimes \alpha) := rk\rho$.

As in the previous section, the product (3.24) allows for a formulation of the extended algebraic structure in perhaps more familiar language. Upon viewing $\mathcal{H}_e$ as $\Omega^*(L) \otimes \hat{\mathcal{E}}$, one can locally expand its elements in the form:

$$\hat{u} = \sum_{k=0}^{3} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \mathbf{U}^{(k)}_{\alpha_1 \ldots \alpha_k}(x) \tag{3.27}$$

where the coefficients $\mathbf{U}_{\alpha_1 \ldots \alpha_k}(x)$ are Grassmann-valued sections of $\mathrm{End}(\mathbf{E})$, i.e. elements of $\hat{\mathcal{E}}$. Then the twisted wedge product (3.24) reads:

$$\hat{u} \boldsymbol{\wedge} \hat{v} = \sum_{k,l=0}^{3} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \wedge dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_l} (\mathbf{U}^{(k)}_{\alpha_1 \ldots \alpha_k}(x) \mathbf{V}^{(l)}_{\beta_1 \ldots \beta_l}(x)) \,, \tag{3.28}$$

where the product $\mathbf{U}^{(k)}_{\alpha_1 \ldots \alpha_k}(x) \mathbf{V}^{(l)}_{\beta_1 \ldots \beta_l}(x)$ *is defined as in (3.3).*

We finally note that one can also define a naive extension of the product $\bullet$ of (2.6) to Grassmann-valued forms with coefficients in $\mathrm{End}(\mathbf{E})$:

$$\hat{u} \bullet \hat{v} = (u \bullet v) \otimes (\hat{f}\hat{g}) = (-1)^{\Delta(\hat{u})rk\hat{v}} \hat{u} \boldsymbol{\wedge} \hat{v} = (-1)^{\Delta(\hat{u})rk\hat{v}+g(\hat{u})\Delta(\hat{v})} \hat{u} \wedge \hat{v} \,, \tag{3.29}$$

where $(\hat{f}\hat{g})$ is again defined as in (3.3). With this definition, one obtains:

$$\hat{u} * \hat{v} = (-1)^{g(\hat{u})rk\hat{v}} \hat{u} \bullet \hat{v} \,. \tag{3.30}$$

3.1.5 The extended differential

The differential on $\Omega^*(L, \mathrm{End}(E))$ extends to $\mathcal{H}_e$ in the obvious manner:

$$d\hat{u} = du \otimes \alpha \,, \tag{3.31}$$

on decomposable elements $\hat{u} = u \otimes \alpha$. The symbol d in the second equality is the differential on $\mathcal{H}$. The differential (3.31) is a graded derivation of $*$, with respect to the total degree:

$$d(\hat{u} * \hat{v}) = (d\hat{u}) * \hat{v} + (-1)^{\deg \hat{u}} \hat{u} * d\hat{v} \,. \tag{3.32}$$

It is also easy easy to check that:

$$\langle d\hat{u}, \hat{v} \rangle_e = -(-1)^{\deg \hat{u}} \langle \hat{u}, d\hat{v} \rangle_e \,. \tag{3.33}$$

3.2 The extended action and restricted odd symplectic form

The data of the previous subsection allows one to write an extended action on the subspace $\mathcal{H}_e^1 = \{\hat{\phi} \in \mathcal{H}_e | \deg \hat{\phi} = \hat{1}\}$ of $\mathcal{H}_e$:

$$S_e(\hat{\phi}) = \int_L \mathrm{str}_e \left[\frac{1}{2} \hat{\phi} * d\hat{\phi} + \frac{1}{3} \hat{\phi} * \hat{\phi} * \hat{\phi} \right] . \tag{3.34}$$

This action was written down in [6] by analogy with the general extension procedure discussed in [21, 22].[5] It is one of the purposes of this paper to show that S_e plays the role of BV action for the string field theory (2.25).

On $\mathcal{H}_e^1$, the product (3.26) agrees with the multiplication $\bullet$ of (2.6) precisely when the worldsheet U(1) charge of the first factor is odd, in which case its Grassmann degree is even (see equation (3.11)). If we extend the evaluation map (3.2) to a map from $\mathcal{H}_e$ to $\mathcal{H}$ defined through:

$$ev_G(u \otimes \alpha) := ev_G(\alpha) u = \alpha_0 u \, , \tag{3.35}$$

we find that:

$$\mathcal{H}^1 = ev_G(M_0) \, , \tag{3.36}$$

where M_0 is the subspace of $\mathcal{H}_e^1$ given by:

$$M_0 = \{\hat{\phi} \in \mathcal{H}_e^1 || \hat{\phi}| = 1\} \, . \tag{3.37}$$

This implies that the restriction of S_e to M_0 is related to the unextended action of (2.25) through:

$$ev_G(S_e(\hat{\phi})) = S(ev_G(\hat{\phi})) \qquad \text{for} \hat{\phi} \in M_0 \, . \tag{3.38}$$

For later reference, we note that the restriction:

$$\omega_0 := \langle ., . \rangle_e |_{\mathcal{H}_e^1 \times \mathcal{H}_e^1} \tag{3.39}$$

of the extended bilinear form to the subspace $\mathcal{H}_e^1$ is an *antisymmetric* nondegenerate bilinear form whose values are Grassmann-odd numbers. This follows from the properties of $\langle \cdot, \cdot \rangle_e$ discussed in the previous section. Let us express ω_0 in terms of the wedge products defined in (3.23) and (3.24). Upon using relation (3.26) and the fact that $\deg \hat{u} = \deg \hat{v} = \hat{1}$, one obtains:

$$\omega_0(\hat{u}, \hat{v}) = (-1)^{(1 - rk\hat{u}) rk\hat{v}} \int_L \mathrm{str}_e(\hat{u} \boldsymbol{\wedge} \hat{v}) \, , \tag{3.40}$$

which in view of the selection rules (3.22) also reads:

$$\omega_0(\hat{u}, \hat{v}) = (-1)^{rk\hat{v}} \int_L \mathrm{str}_e(\hat{u} \boldsymbol{\wedge} \hat{v}) = (-1)^{rk\hat{v} + g(\hat{u})\Delta(\hat{v})} \int_L \mathrm{str}_e(\hat{u} \wedge \hat{v}) \, . \tag{3.41}$$

Moreover, we notice that $\omega_0(\hat{u}, \hat{v})$ vanishes unless $g(\hat{u}) + g(\hat{v})$ is odd (this follows from the constraints $\deg \hat{u} = \deg \hat{v} =$ *odd* and the selection rule (3.21)); this establishes that ω_0 takes Grassmann-odd values. These observations will be useful in section 7.

[5] As the latter translates in our conventions and with certain modifications.

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3.3 Superspace formulation of the extended action

The extended action (3.34) can be formulated in superspace language as follows. Consider the supermanifold $\mathcal{U} := \Pi TL$ obtained by applying parity reversal on the fibers of the tangent bundle of L. Superfunctions defined on $\mathcal{L}$ are (Grassmann-valued) superfields on L, with odd superspace coordinates θ^j associated with the tangent vectors $\partial_j = \partial/\partial x^j$ defined by a coordinate system $\{x^j\}_{j=1\dots 3}$ on L.

Grassmann-valued forms with coefficients in End($\mathbf{E}$) can be put into correspondence with superfields valued in End($\mathbf{E}$) upon identifying $\hat{u}$ of equation (3.27) with:

$$\mathbf{U}(x,\theta) = \sum_{k=0}^{3} \theta^{\alpha_1} \cdots \theta^{\alpha_k} \mathbf{U}^{(k)}_{\alpha_1 \dots \alpha_k}(x) \,. \tag{3.42}$$

In order to translate the action (3.34) in superspace language, one must use the somewhat unusual convention that the components $\mathbf{U}^{(k)}_{\alpha_1\dots\alpha_k}(x) \in \hat{\mathcal{E}}$ of superfields of the form (3.42) are multiplied with the product (3.3) and that the sign obtained when commuting θ^j with such a component is $(-1)^{\sigma(\mathbf{U})}$. With these conventions, one can check [6] that multiplication of superfields reproduces the product (3.26) for the associated forms, which allows one to write the extended action as:

$$S_e(\mathbf{\Phi}) = \int_L d^3x \int d^3\theta \, \mathrm{str}_e \left[\frac{1}{2} \mathbf{\Phi} \mathbf{D} \mathbf{\Phi} + \frac{1}{3} \mathbf{\Phi}\mathbf{\Phi}\mathbf{\Phi} \right] , \tag{3.43}$$

where $\mathbf{\Phi}$ is the superfield associated with $\hat{\phi}$ and $\mathbf{D} = \theta^j \partial_j$. In this paper, we shall only use the differential form language of (3.34).

3.4 The underlying superbundle and the physical role of $\mathbb{Z}$-grading

It is clear from our construction that the extended boundary product $*$ (and thus the extended action (3.34)) depend only on the mod two reduction of the relative D-brane grade Δ. To formalize this, let us consider the reduction $E = E_{\mathrm{even}} \oplus E_{\mathrm{odd}}$ of the $\mathbb{Z}$-grading of $\mathbf{E}$, where:

$$E_{\mathrm{even}} = \oplus_{n=\mathrm{even}} E_n \,, \qquad E_{\mathrm{odd}} = \oplus_{n=\mathrm{odd}} E_n \,. \tag{3.44}$$

This allows us to view $\mathbf{E}$ as a superbundle [20], while forgetting the finer data associated with the $\mathbb{Z}$-grading. It is clear that the boundary product, extended boundary product and extended action depend only on this superbundle structure. In particular, one has only two classes of extended actions. The first corresponds to the case $E_{\mathrm{even}} = 0$ or $E_{\mathrm{odd}} = 0$ and can be recognized as the extended Chern-Simons action coupled to the bundle $\mathbf{E} = E_{\mathrm{odd}}$ or $\mathbf{E} = E_{\mathrm{even}}$. The second corresponds to $E_{\mathrm{even}} \neq 0$ and $E_{\mathrm{odd}} \neq 0$ and can be viewed as an extended version of the 'supergroup Chern-Simons action' [23] coupled to $\mathbf{E} = E_{\mathrm{odd}} \oplus E_{\mathrm{even}}$. This agrees with ideas proposed in [14].

The D-brane grade plays the role of specifying the finer $\mathbb{Z}$-grading given by the world-sheet U(1) charge. It is this piece of data which defines the subspace $\{\hat{\phi} \in \mathcal{H}^1_e \,|\, |\hat{\phi}| = 1\}$ on which the extended action (3.34) reduces to the unextended functional (2.25). As we shall see in the next section, the extended theory can be viewed as a classical BV system, with S_e

playing the role of tree-level master action. From the BV perspective, the choice of D-brane grading is what specifies both the BV ghost number and the so-called *classical gauge*. In particular, two theories which have the same underlying superbundle but distinct choices of $\mathbb{Z}$-grading have the same tree-level BV actions but correspond to different choices for these two pieces of data. Since the ghost grading is physically relevant (in particular, as we recall at the end of section 4.1.3, it specifies the algebra of classical on-shell gauge-invariant observables, given the other tree-level BV data), different choices of D-brane grading lead to *different* physical theories, in spite of the fact that the difference may not be manifest in the BV action itself. This is the crucial conceptual distinction between our approach and the proposals of [14].

4. The extended action as a classical master action

In this section we show that the extended action satisfies the classical master equation with respect to a BV bracket induced by an odd symplectic form associated to the extended bilinear form. We also show that S_e reduces to the unextended action S in a certain 'classical gauge'. These extremely general results are valid for an arbitrary collection of graded branes (of distinct grades) wrapping the cycle L, and hold for any topology of the cycle and background connection.

Our approach uses a certain variant of the geometric BV framework developed in [24, 25, 26] and [15, 27, 28, 29, 30]. This formalism has the advantage that it is computationally compact and well-adapted to topologically-nontrivial situations. In fact, it turns out that the current version of the geometric BV formalism is incomplete, and we shall have to extend it in an appropriate manner. The problem is that the geometric description presented in the references just cited does not keep track of the BV ghost number. Indeed, the geometric formalism is usually discussed in terms of a P-manifold, i.e. a supermanifold endowed with an odd symplectic form. While this correctly considers the Grassmann parity of various BV fields, it fails to account for the ghost grading. This $\mathbb{Z}$-grading on the space of superfunctions plays a crucial role in the bottom-up (or homological) approach to BV quantization [31]–[35] and in many questions of direct physical significance. For example, it is a central result of the BV formalism that the BRST cohomology in *ghost* degree zero computes the space of on-shell gauge-invariant observables of the system. Since the current geometric formulation does not consider the ghost grading, it does not allow for a description of this (and other) fundamental results. In particular, two *distinct* BV systems can have the same supermanifold interpretation and the same BV action, so they cannot be distinguished by the current geometric approach. This can lead to confusion when applied to our models. To avoid such problems, one must refine the geometric formulation by explicitly including the ghost grading. This can be done with the help of *$\mathbb{Z}$-graded supermanifolds*, which were recently discussed in [16]. We start with a brief account of graded supermanifolds and continue by presenting a $\mathbb{Z}$-graded version of the geometric BV formalism. We then apply it to our theories in order to obtain a complete description of the associated BV systems. Most of this section is formulated in an entirely general manner, and may be of independent interest for foundational studies of BV quantization.

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4.1 Covariant description of classical BV systems in the graded supermanifold approach

4.1.1 Supermanifold conventions

We remind the reader that there are two major proposals for a rigorous definition of supermanifolds, the so-called Berezin-Konstant [36] and De Witt-Rogers [37, 38] theories. The major difference between the two is that the definition of a De Witt-Rogers supermanifold requires the choice of an auxiliary Grassmann algebra G, the 'algebra of constants'. Berezin's approach is based on an 'intrinsic' sheaf of superalgebras, which leads to a formulation in terms of ringed spaces ('superschemes'). In this theory, the manifold has only even points, while the odd coordinates appear as a form of 'algebraic fuzz'. By contrast, the De Witt-Rogers theory constructs supermanifolds which possess both even and odd points, thus leading to a geometrization of the odd directions; this geometric description of odd coordinates depends on the algebra of constants G. It is a basic result that the set of G-valued points (defined in a manner similar to that employed in scheme theory) of a Berezin supermanifold defines a De Witt-Rogers supermanifold[6] (though not every De Witt-Rogers supermanifold can be obtained in this manner [38]). Since the extended theory (3.34) incorporates the auxiliary Grassmann algebra G, we shall employ the formalism due to De Witt and Rogers. Thus all of our supermanifolds are understood in the De Witt-Rogers sense.[7]

Given a (complex) De Witt-Rogers supermanifold M (modeled over the algebra of constants G), its tangent space at a point p is a super-bimodule T_pM over G (see appendix A), whose left and right module structures are compatible:

$$\alpha X_p = (-1)^{\epsilon_\alpha \epsilon(X_p)} X_p \alpha \,, \tag{4.1}$$

for α an element of G and X_p an element of T_pM. We make the convention that ϵ denotes the $\mathbb{Z}_2$-degree of an element in the space to which it belongs. Thus ϵ_α is the Grassmann parity of α, while $\epsilon(X_p)$ is the parity of X_p with respect to the $\mathbb{Z}_2$-grading on T_pM. The disjoint union of the tangent spaces gives the tangent bundle TM.

Globally defined G-valued functions on M form a commutative $\mathbb{Z}_2$-graded ring $\mathcal{F}(M, G)$ with respect to pointwise multiplication, with:

$$\epsilon_F = \hat{0} \quad \text{if } F(p) \in G_0 \text{for all } p \in M \,, \qquad \epsilon_F = \hat{1} \quad \text{if } F(p) \in G_1 \text{for all } p \in M \,. \tag{4.2}$$

This is also a left- and right- $\mathbb{Z}_2$-graded algebra over the ring of constants G. Left and right derivations of this algebra give so-called left and right vector fields on M (this is discussed in more detail in appendix B). The spaces of left/right vector fields are graded in the obvious manner, with even and odd derivations corresponding to even and odd vector fields. It is customary to identify left and right derivations, and we shall do so in the following (see appendix B for details of this construction). This allows us to speak simply

[6] More precisely, it defines a so-called H^∞-supermanifold [38].

[7] The formalism we use can in fact be applied to so-called G^∞-supermanifolds, which are a generalization of H^∞-supermanifolds [38].

about vector fields. With this convention, a vector field X can act both to the left (as a left derivation) and to the right (as a right derivation), with the two actions related by applying the sign rule. We shall indicate the left and right actions by superscript arrows pointing respectively to the right and left. For every function F we thus have:

$$F \overleftarrow{X} = (-1)^{\epsilon_F \epsilon_X} \overrightarrow{X} F := dF(X)\,, \tag{4.3}$$

where dF is by definition the differential of F. This is a complex-linear function defined on the space of vector fields, which is also G-linear in the obvious sense:

$$dF(X\alpha) = dF(X)\alpha\,, \qquad dF(\alpha X) = (-1)^{\epsilon_\alpha \epsilon_F} \alpha dF(X)\,. \tag{4.4}$$

It induces G-linear functionals d_pF on each of the tangent spaces T_pM.

The space of vector fields is endowed with a Lie bracket, which in terms of the action on functions is given by:

$$F \overleftarrow{[X,Y]} := -F(\overleftarrow{X}\overleftarrow{Y} - (-1)^{\epsilon_X \epsilon_Y} \overleftarrow{Y}\overleftarrow{X}) \Longleftrightarrow \overrightarrow{[X,Y]} F = (\overrightarrow{X}\overrightarrow{Y} - (-1)^{\epsilon_X \epsilon_Y} \overrightarrow{Y}\overrightarrow{X})F\,. \tag{4.5}$$

This operation satisfies $\epsilon_{[X,Y]} = \epsilon_X + \epsilon_Y$ and is graded-symmetric and G-bilinear:

$$[X,Y] = -(-1)^{\epsilon_X \epsilon_Y} [Y,X]\,, \qquad [\alpha X, Y\beta] = \alpha\, [X,Y]\, \beta\,. \tag{4.6}$$

It also satisfies the graded Jacobi identity:

$$[[X,Y],Z] + (-1)^{\epsilon_X(\epsilon_Y + \epsilon_Z)}[[Y,Z],X] + (-1)^{\epsilon_Z(\epsilon_X+\epsilon_Y)}[[Z,X],Y] = 0\,. \tag{4.7}$$

Endowed with this commutator, the space of vector fields becomes a $\mathbb{Z}_2$-graded Lie algebra (Lie superalgebra).

The space of functionals $\eta(X)$ obeying G-linearity constraints of the type (4.4) forms a $\mathbb{Z}_2$-graded G-bimodule in the obvious manner. This is the space $\Omega^1(M)$ of one-forms on M. One defines higher rank forms with the help of the wedge product:

$$\rho \wedge \eta = \rho \otimes \eta - (-1)^{\epsilon_\rho \epsilon_\eta} \eta \otimes \rho\,, \tag{4.8}$$

which has the graded symmetry property:

$$\rho \wedge \eta = (-1)^{\epsilon_\rho \epsilon_\eta + 1} \eta \wedge \rho\,. \tag{4.9}$$

Upon taking iterated wedge products one obtains forms of arbitrary ranks and (4.8) extends to such forms in the obvious fashion. One also has an exterior differential, obtained by extending (4.3). In particular, a two-form $\omega(X,Y)$ on X is a G-valued complex-bilinear functional on vector fields which has the properties:

$$\begin{aligned} \epsilon_{\omega(X,Y)} &= \epsilon_X + \epsilon_Y + \epsilon_\omega \\ \omega(\alpha X, Y\beta) &= (-1)^{\epsilon_\alpha \epsilon_\omega} \alpha \omega(X,Y)\beta \\ \omega(X,Y) &= (-1)^{\epsilon_X \epsilon_Y + \epsilon_\omega} \omega(Y,X)\,. \end{aligned} \tag{4.10}$$

The quantity $\epsilon_\omega \in \{\hat{0},\hat{1}\}$ defines its parity: ω is even if $\epsilon_\omega = \hat{0}$ and odd if $\epsilon_\omega = \hat{1}$. A two-form ω is called *symplectic* if it is nondegenerate and closed ($d\omega = 0$).

Local coordinates give independent Grassmann-valued functions z^a defined on an open subset of M, whose parities we denote by $\epsilon_a := \epsilon(z_a)$. Given such coordinates, one has locally-defined vector fields $\partial^l_a = (-1)^{\epsilon_a}\partial^r_a$ (of parity ϵ_a), which are uniquely determined by:

$$\overrightarrow{\partial^l_a}\, z^b = z^b \overleftarrow{\partial^r_a} = \delta^b_a\,. \tag{4.11}$$

Their action on a function F defines its left and right derivatives:

$$\overrightarrow{\partial^l_a}\, F = \frac{\partial_l F}{\partial z^a} = (-1)^{\epsilon_F\epsilon_a} dF(\partial^l_a)\,, \qquad F \overleftarrow{\partial^r_a} = \frac{\partial_r F}{\partial z^a} = dF(\partial^r_a)\,. \tag{4.12}$$

For the coordinate functions one obtains:

$$dz^a(\partial^r_b) = (-1)^{\epsilon_a} dz^a(\partial^l_b) = \delta^a_b\,. \tag{4.13}$$

This allows us to write dF in the form:

$$dF = \frac{\partial_r F}{\partial z^a}dz^a = dz^a\frac{\partial_l F}{\partial z^a} = (-1)^{\epsilon_a(\epsilon_F+1)}\frac{\partial_l F}{\partial z^a}dz^a\,. \tag{4.14}$$

Given a vector field X, one can expand it locally as:

$$X = X^a_l\partial^l_a = \partial^r_a X^a_r\,, \tag{4.15}$$

which defines its left and right coefficients X^a_l and X^a_r (=locally defined Grassmann-valued functions). Equation (4.14) then gives:

$$dF(X) = \frac{\partial_r F}{\partial z^a}X^a_r = (-1)^{\epsilon_X\epsilon_F}X^l_a\frac{\partial_l F}{\partial z^a}\,. \tag{4.16}$$

Let us next consider the local expression of an *odd* symplectic form ω. If one defines its coefficients through:

$$\omega_{ab} := \omega(\partial^r_a,\partial^r_b) = (-1)^{\epsilon_a+\epsilon_b}\omega(\partial^l_a,\partial^l_b) = (-1)^{\epsilon_a}\omega(\partial^l_a,\partial^r_b) = (-1)^{\epsilon_b}\omega(\partial^r_a,\partial^l_b)\,, \tag{4.17}$$

then it is easy to check that:

$$\omega = -\frac{1}{2}\omega_{ab}dz^b\wedge dz^a \tag{4.18}$$

(note the reversed order in the wedge product). Its value on an arbitrary pair of vector fields then follows from the bi-linearity property listed in (4.10):

$$\omega(X,Y) = (-1)^{(\epsilon_X+\epsilon_a)\epsilon_b}\omega_{ab}X^a_rY^b_r\,. \tag{4.19}$$

Definition (4.17) and relations (4.10) imply the properties:

$$\epsilon(\omega_{ab}) = \epsilon_a+\epsilon_b+1\,, \qquad \omega_{ab} = -(-1)^{\epsilon_a\epsilon_b}\omega_{ba}\,. \tag{4.20}$$

4.1.2 $\mathbb{Z}$-graded supermanifolds

The collection $\mathcal{F} = (\mathcal{F}(U,G))$ of G-valued functions defined on open subsets U of G forms a sheaf of superalgebras with respect to the $\mathbb{Z}_2$-grading given by ϵ. A $\mathbb{Z}$-graded supermanifold [16] is a supermanifold endowed with a $\mathbb{Z}$-grading s on this sheaf. This $\mathbb{Z}$-grading is required to be compatible with pointwise multiplication $s(FG) = s(F) + s(G)$ and with restriction from an open set to its open subsets. We also require $s(\alpha F) = s(F\alpha) = s(F)$ for $\alpha \in G$. The $\mathbb{Z}_2$-grading ϵ need not be the mod 2 reduction of s; in fact, this is almost never the case if one works with De Witt-Rogers supermanifolds.[8]

A $\mathbb{Z}$-grading on $\mathcal{F}$ can be specified by giving an atlas $\{(U, z^a_U)\}$ of local coordinates and picking integer grades s^U_a for z^U_a such that the change of coordinates from z^U_a to z^V_a (when U intersects V) is compatible with these degrees. For simplicity, let us restricts the elements of $\mathcal{F}(U,G)$ to be functions which are polynomial in coordinates.[9] Such a function has the form:

$$F(p) = \sum_{k=1}^{N} \sum_{a_1 \dots a_k} \alpha_{a_1 \dots a_k} z_U^{a_1}(p) \cdots z_U^{a_k}(p) \Leftrightarrow F = \sum_{k=1}^{N} \sum_{a_1 \dots a_k} \alpha_{a_1 \dots a_k} z_U^{a_1} \cdots z_U^{a_k}\,, \tag{4.21}$$

where $\alpha_{a_1 \dots a_k}$ are elements of G and the sum runs over monomials of degree smaller than some positive integer N. We extend s_a to a $\mathbb{Z}$-grading on $\mathcal{F}(U,G)$ by declaring that $s(z^{a_1} \cdots z^{a_k}) = s_{a_1} + \cdots + s_{a_k}$. A function (4.21) is s-homogeneous of degree σ if all of the monomials appearing in its expansion satisfy $s(z^{a_1} \cdots z^{a_k}) = \sigma$. It is clear that this grading is compatible with pointwise multiplication and restriction to open subsets of U, and satisfies $s(\alpha F) = s(F\alpha) = s(F)$.

If V is another coordinate neighborhood in the distinguished atlas (such that U intersects V), then on the intersection $U \cap V$ one can express z^a_V as:

$$z^a_V = z^a_V(z_U)\,, \tag{4.22}$$

where we assume that the transition functions are polynomial. The compatibility condition requires that s_a coincide with the degree of the function $z^a_V(z_U)$, defined with respect to the coordinates z^U. This assures us that the degree of a function in $\mathcal{F}(U \cap V, G)$ does not depend on the coordinates in which it is computed, and thus we have a well-defined $\mathbb{Z}$-grading on the sheaf $\mathcal{F}$.[10]

Notice that the Grassmann coefficients α play no role the grading s, i.e. one can formally write $s(\alpha_{a_1 \dots a_k}) = 0$. As mentioned above, the Grassmann grading ϵ_F of a function F need not coincide with the mod 2 reduction of its $\mathbb{Z}$-grading. For example, if $F = \alpha z^{a_1} \cdots z^{a_k}$, then $\epsilon_F = \epsilon_\alpha + \epsilon_{a_1} + \cdots + \epsilon_{a_k}$, but $s_F = s_{a_1} + \cdots + s_{a_k}$, so that s_F (mod 2) may differ from ϵ_F even if one chooses s_a such that $s_a(mod\ 2) = \epsilon_a$. This mismatch between

[8]The reason is that ϵ must satisfy $\epsilon(F\alpha) = \epsilon_F + \epsilon_\alpha$, while s satisfies $s(F\alpha) = s(F)$. Hence $\epsilon = s$ (mod 2) would require $\epsilon_\alpha = 0$ for all $\alpha \in G$, which is only possible if G has no odd generators.

[9]One can also consider formal power series, which gives a formal $\mathbb{Z}$-graded supermanifold. If one wishes to extend this beyond formal power series, one has to deal with issues of convergence, which we wish to avoid.

[10]To globalize this argument one needs to assume the existence of an appropriate partition of unity etc.

$\mathbb{Z}_2$-grading and $\mathbb{Z}$-grading is due to the presence of the Grassmann algebra of constants G, and thus is an inescapable feature of working with De Witt-Rogers supermanifolds. The $\mathbb{Z}$ and $\mathbb{Z}_2$-gradings s and ϵ must be viewed as independent pieces of data.

The integer grading on Grassmann-valued functions allows us to introduce $\mathbb{Z}$-gradings on the spaces of vector fields and differential forms. Since vector fields X are complex-linear maps from $\mathcal{F}$ to itself, we shall say that X is s-homogeneous of degree σ if:

$$s(F \overleftarrow{X}) = s(F) + s_X \tag{4.23}$$

for some integer s_X which defines the s-degree of X. It is clear that $s([X,Y]) = s_X + s_Y$.

We shall say that a local coordinate system is *s-homogeneous* if the associated coordinate functions z^a are s-homogeneous elements of $\mathcal{F}$. In this case, we denote $s(z^a)$ by s_a. Given an s-homogeneous coordinate system, it is clear that $s(F \overleftarrow{\partial^r_a}) = s(\overrightarrow{\partial^l_a} F) = s(F) - s_a$, which implies:

$$s(\partial^r_a) = s(\partial^l_a) = -s_a\,. \tag{4.24}$$

This allows us to introduce a $\mathbb{Z}$-grading on the tangent spaces T_pM by using $X_p = (\partial^r_a)_p X^r_a(p)$ and the rule $s(X^r_a(p)) = 0$ (since $X^a_r(p)$ is a Grassmann constant, i.e. an element of G). It is clear that this grading is independent of the choice of s-homogeneous coordinates; it can be defined more invariantly by considering localization of vector fields. This $\mathbb{Z}$-grading, as well as the $\mathbb{Z}$-grading (4.23) on vector fields, have no direct relation to the $\mathbb{Z}_2$-grading ϵ.

If η is a linear functional on vector fields, then we define its s-degree through:

$$s(\eta(X)) = s(X) + s_\eta\,. \tag{4.25}$$

In s-homogeneous coordinates, the relation $dz^a(\partial^r_b) = \delta^a_b$ implies:

$$s(dz^a) = s(z^a) = s_a\,. \tag{4.26}$$

Moreover, we obtain:

$$s(dF) = s_F\,, \qquad \text{i.e.} \qquad s(dF(X)) = s_F + s_X\,, \tag{4.27}$$

and $s(\partial_l F/\partial z^a) = s(\partial_r F/\partial z^a) = s_F - s_a$.

This grading extends to multilinear forms in the obvious manner. For a two-form, we have:

$$s(\omega(X,Y)) = s(X) + s(Y) + s_\omega \tag{4.28}$$

(notice that $\omega(X,Y)$ is an element of $\mathcal{F}(M,G)$). In s-homogeneous local coordinates, this gives:

$$s(\omega_{ab}) = s_\omega - s_a - s_b\,, \tag{4.29}$$

where ω_{ab} is the function $z \to \omega_{ab}(z)$.

4.1.3 Basics of the geometric framework

We now give a brief outline of a $\mathbb{Z}$-graded version of the geometric BV formalism. This is a supergeometric version of the symplectic formalism of hamiltonian mechanics, endowed with the supplementary data of an integer-valued grading (the ghost grading). Its starting point is a *graded P-manifold*, i.e. a graded De Witt-Rogers supermanifold M (modeled[11] on $\mathbb{R}^{n,n}$), endowed with an *odd* symplectic form ω which is s-homogeneous of degree $s_\omega = -1$

Given a P-manifold, the odd symplectic form allows one to define a (right) hamiltonian vector field Q_F associated with an arbitrary (Grassmann-valued) function F on M:

$$dF(X) = \omega(Q_F, X)\,. \tag{4.30}$$

This equation is sensible since both the left and right hand sides are linear with respect to the right G-module structure on vector fields; non-degeneracy of ω assures the existence of a unique solution.

It is clear from this definition that:

$$Q_{\alpha F} = (-1)^{\epsilon_\alpha}\alpha Q_F\,, \qquad Q_{F\alpha} = Q_F\alpha\,, \tag{4.31}$$

for any Grassmann constant α.

Since the symplectic form satisfies $\epsilon_\omega = \hat{1}$ and $s_\omega = -1$, the vector field Q_F has parity $\epsilon_{Q_F} = \epsilon_F + \hat{1}$ and ghost number $s_{Q_F} = s_F + 1$. In local coordinates z^a, one has the expansions $Q_F = Q^a_{F,l}\partial^l_a = \partial^r_a Q^a_{F,r}$, with the components:

$$Q^a_{F,l} = (-1)^{\epsilon_F+\epsilon_a+1}\frac{\partial_l F}{\partial z^b}\omega^{ba}\,, \qquad Q^a_{F,r} = (-1)^{\epsilon_a+1}\omega^{ab}\frac{\partial_r F}{\partial z^b}\,, \tag{4.32}$$

where ω^{ab} is the inverse of the matrix ω_{ab}:

$$\omega^{ab}\omega_{bc} = \omega_{cb}\omega^{ba} = \delta^a_c\,. \tag{4.33}$$

Note the properties:

$$\epsilon(\omega_{ab}) = \epsilon_a + \epsilon_b + 1\,, \qquad s(\omega^{ab}) = s_a + s_b + 1\,, \qquad \omega^{ab} = -(-1)^{(\epsilon_a+1)(\epsilon_b+1)}\omega_{ba}\,. \tag{4.34}$$

Given two functions F, G on M,[12] we define their *odd Poisson bracket* (antibracket) through:

$$\{F,G\} = -\omega(Q_F,Q_G) = -dF(Q_G) = (-1)^{(\epsilon_F+1)(\epsilon_G+1)}dG(Q_F) = \frac{\partial_r F}{\partial z^a}\omega^{ab}\frac{\partial_l G}{\partial z^b}\,. \tag{4.35}$$

[11]In our case n will be infinite, as always in field theory. We shall neglect the well-known problems with infinite-dimensional supermanifolds (see, for example, [39]). In fact, we shall later apply this formalism to linear supermanifolds only, for which the treatment can be made rigorous in terms of Banach supermanifolds. The condition 'modeled on $\mathbb{R}^{n,n}$' means that one has an equal number of even and odd coordinates; in our application, this can be formulated in terms of countable coordinate frames.

[12]We sometimes use the symbol G to denote a Grassmann-valued function on M. This should not be confused with the underlying Grassmann algebra, which is denoted by the same letter.

One has $\epsilon_{\{F,G\}} = \epsilon_F + \epsilon_G + 1$ and $s(\{F,G\}) = s_F + s_G + 1$. It is easy to check the properties:

$$\{FG,H\} = F\{G,H\} + (-1)^{\epsilon_F\epsilon_G} G\{F,H\}\,, \qquad \{F,GH\} = \{F,G\}H + (-1)^{\epsilon_G\epsilon_H}\{F,H\}G$$
$$\{\alpha F, G\beta\} = \alpha\{F,G\}\beta\,, \qquad \{F,G\} = -(-1)^{(\epsilon_F+1)(\epsilon_G+1)}\{G,F\}. \quad (4.36)$$

as well as the odd graded Jacobi identity:

$$\{\{F,G\},H\} + (-1)^{(\epsilon_F+1)(\epsilon_G+\epsilon_H)}\{\{G,H\},F\} + (-1)^{(\epsilon_H+1)(\epsilon_F+\epsilon_G)}\{\{H,F\},G\}\,. \quad (4.37)$$

In particular, the space of G-valued functions on M forms an odd Lie superalgebra[13] with respect to the antibracket. Equation (4.35) shows that:

$$F \overleftarrow{Q}_G = -\{F,G\} \Longleftrightarrow \overrightarrow{Q}_G\, F = -(-1)^{\epsilon_F(\epsilon_G+1)}\{F,G\}\,. \quad (4.38)$$

Together with the Jacobi identity, this implies:

$$Q_{\{F,G\}} = [Q_F, Q_G]\,. \quad (4.39)$$

Thus the map $F \rightarrow Q_F$ acts as an 'odd morphism' (the composition of a morphism of $\mathbb{Z}_2$-graded Lie algebras with parity change). This translates the odd Lie superalgebra language appropriate for functions into the $\mathbb{Z}_2$-graded Lie algebra language relevant for vector fields.

In the context of BV quantization, the antibracket is interpreted as the BV bracket. For any function F, one has:[14]

$$\{F,F\} = -\omega(Q_F, Q_F) = -dF(Q_F)\,, \qquad Q_{\{F,F\}} = [Q_F, Q_F]\,. \quad (4.40)$$

Hence given an action (even function of ghost degree zero) S_{BV} on our supermanifold, the classical master equation can be written in the equivalent forms:

$$\{S_{BV}, S_{BV}\} = 0 \Leftrightarrow QS_{BV} = 0 \Leftrightarrow \omega(Q,Q) = 0 \Leftrightarrow [Q,Q] = 0 \Leftrightarrow Q^2(F) = 0 \quad \text{for all } F\,, \quad (4.41)$$

where we defined $Q := Q_{S_{BV}}$. It is clear that $\epsilon(Q_F) = \hat{1}$ and $s(Q_F) = +1$. In particular, a classical BV system defines a so-called *QP-manifold* [15], i.e. a P-manifold endowed with an odd nilpotent vector Q field which preserves the odd symplectic form.

We remind the reader that an odd vector field Q on a supermanifold is called *nilpotent* (or *homological*) if it satisfies $[Q,Q] = 0 \Leftrightarrow Q^2F = 0$ for all F. Given such a vector field, the space $\mathcal{F}(M,G)$ of globally defined Grassmann-valued functions (viewed as a complex vector space) becomes a complex with respect to the differential Q. If the underlying supermanifold is $\mathbb{Z}$-graded, and if Q has $\mathbb{Z}$-degree equal to $+1$, then $(\mathcal{F}(M,G), Q)$ is a $\mathbb{Z}$-graded cochain complex. The main result of the bottom-up approach to BV quantization is that, for $Q = Q_{BV}$, the cohomology of this complex in integer degree zero computes

[13] An odd Lie superalgebra [40] is simply the parity change of a Lie superalgebra. This is obtained by reversing the parity of all elements, while leaving the Lie bracket unchanged. Together with pointwise multiplication of functions, the antibracket endows the space $\mathcal{F}(M,G)$ with the structure of a so-called *odd Poisson algebra* or *Gerstenhaber algebra*.

[14] Note that (4.35) implies $\omega(Q_F, Q_F) = -dF(Q_F) = 0$ if F is odd.

the space of gauge-invariant functionals on the shell of the associated classical action (the relation between the BV action and the classical action is described in geometric terms in the next section). Since the space of such observables has direct physical meaning, it is clear that two BV systems which have different ghost gradings (but the same underlying manifold and odd symplectic form) must be considered as distinct. Otherwise, there would be no clear way of recovering the classical data from the geometric formalism — in particular, one would reach the paradox that two classical systems with very different algebras of on-shell gauge-invariant observables are equivalent, provided that they differ 'only' by the choice of ghost grading, a statement which is clearly incorrect. This observation justifies the need for a $\mathbb{Z}$-graded geometric formalism, and is crucial for a correct understanding of graded D-brane systems. We believe that a correct geometric description of BV systems must systematically consider the ghost grading. Below, we limit ourselves to re-formulating some basic results of the geometric framework (which will be needed in our application) in the graded manifold language.

4.1.4 Gauges and BRST transformations

In the geometric formalism, a *gauge* corresponds to the choice of a lagrangian sub-supermanifold of M, i.e. a sub-supermanifold $\mathcal{L}$ whose total dimension is half of the total dimension of M[15] and with the property that ω restricts to zero on $\mathcal{L}$. To make contact with the bottom-up approach, one must also assume that $\mathcal{L}$ is *s-homogeneous*, i.e. its tangent bundle $T\mathcal{L}$ decomposes as a direct sum of s-homogeneous subbundles of $TM|_{\mathcal{L}}$:

$$T\mathcal{L} = \oplus_{s\geq 0} TM|_{\mathcal{L}}(s)\,, \tag{4.42}$$

where $TM|_{\mathcal{L}}(s)$ is the subbundle of $TM|_{\mathcal{L}}$ consisting of elements of ghost degree equal to s.[16]

The path integral in this gauge is given by integrating $e^{-\frac{i}{\hbar}S_{BV}}|_{\mathcal{L}} = e^{-\frac{i}{\hbar}S_{\mathcal{L}}}$ along $\mathcal{L}$, where $S_{\mathcal{L}} := S_{BV}|_{\mathcal{L}}$ is the restriction of S_{BV} to $\mathcal{L}$. This is a global version of the usual description of gauges in terms of fields and antifields and gauge-fixing fermions. We shall follow common practice and omit the word 'super' when talking about submanifolds of a supermanifold. We remind the reader that an odd vector field on M can be viewed as an odd section of TM or an even section of the parity changed bundle ΠTM. It is sometimes convenient to work with even sections only, in which case the parity of a vector field is made clear by the presence or absence of parity change on the underlying bundle. We shall sometimes use this convention in what follows. Therefore, a section of a bundle will always mean an even section unless explicitly stated otherwise.

Let us recall from [15] how the BRST transformations of the gauge-fixed action are realized in the geometric formalism. Choosing a gauge $\mathcal{L}$, one constructs a symmetry of $S_{\mathcal{L}}$ as follows. Upon restricting Q to $\mathcal{L}$, one obtains a section of the bundle $\Pi TM|_{\mathcal{L}}$. In order

[15] If a supermanifold is modeled on $\mathbb{R}^{p|q}$, then its total dimension is $p+q$.

[16] It is easy to see that $TM = T_+M \oplus T_-M$ (where $T_\pm M = \oplus_{\pm s\geq 0} T(s)$) gives a lagrangian decomposition of TM; this follows from the condition $s_\omega = -1$. An s-homogeneous lagrangian submanifold is an integral submanifold for the Frobenius distribution T_+M. This distribution is clearly integrable, since $s([X,Y]) = s(X) + s(Y) \geq 0$ for all vector fields X, Y satisfying $s(X), s(Y) \geq 0$.

to produce an (odd) vector field on $\mathcal{L}$, one considers the decomposition $TM|_{\mathcal{L}} = T\mathcal{L} \oplus N\mathcal{L}$, where:

$$N\mathcal{L} = \oplus_{s<0} TM|_{\mathcal{L}}(s)\,. \tag{4.43}$$

It is clear from the condition $s_\omega = -1$ that this is a lagrangian splitting of the restricted tangent bundle $TM|_{\mathcal{L}}$, i.e. ω vanishes on $T\mathcal{L} \times T\mathcal{L}$ and $N\mathcal{L} \times N\mathcal{L}$ and is non-degenerate on $T\mathcal{L} \times N\mathcal{L}$ and on $N\mathcal{L} \times T\mathcal{L}$. As explained in more detail below, this decomposition of $TM|_{\mathcal{L}}$ is related to the field-antifield split of the local formalism.

One has a similar decomposition $\Pi TM|_{\mathcal{L}} = \Pi T\mathcal{L} \oplus \Pi N\mathcal{L}$ of the parity changed bundle. If T, R are the associated projectors of $\Pi TM|_{\mathcal{L}}$ onto $\Pi T\mathcal{L}$ and $\Pi N\mathcal{L}$, then the relations:

$$q := TQ|_{\mathcal{L}}\,, \qquad q^* := RQ|_{\mathcal{L}} \tag{4.44}$$

define sections of $\Pi T\mathcal{L}$ and $\Pi N\mathcal{L}$ which give a decomposition of Q on $\mathcal{L}$:

$$Q|_{\mathcal{L}} = q + q^*\,. \tag{4.45}$$

It is clear that the operators T and R are s-homogeneous of degree zero, so that the vector fields q and q^* are s-homogeneous of degree $+1$. More generally, vectors $u \in TM|_{\mathcal{L}}$ decompose as $u = \mathbf{u} + \mathbf{u}^*$, with $\mathbf{u} \in T\mathcal{L}$ and $\mathbf{u}^* \in N\mathcal{L}$. Upon using this in the defining relation for Q, one obtains:

$$dS_{BV}(u) = \omega(u, Q) = \omega(\mathbf{u}, q^*) + \omega(\mathbf{u}^*, q)\,. \tag{4.46}$$

which combines with $dS_{BV}(u) = dS_{BV}(\mathbf{u}) + dS_{BV}(\mathbf{u}^*) = dS_{\mathcal{L}}(\mathbf{u}) + dS_{BV}(\mathbf{u}^*)$ to give:

$$dS_{\mathcal{L}}(\mathbf{u}) = \omega(\mathbf{u}, q^*)\,, \qquad dS_{BV}(\mathbf{u}^*) = \omega(\mathbf{u}^*, q)\,. \tag{4.47}$$

The second equation shows that the first order term in the Taylor expansion of S_{BV} in antifields is proportional with q. The first implies that the value of q^* at a point p in $\mathcal{L}$ vanishes precisely when p is critical for $S_{\mathcal{L}}$. Hence the critical set of $S_{\mathcal{L}}$ is the locus where Q is tangent to $\mathcal{L}$. Combining this with equation (4.45) and using the nilpotence of Q shows that q squares to zero 'on the shell of $S_{\mathcal{L}}$':

$$[q, q] = 0 \quad \text{on} \qquad \mathrm{Crit}(S_{\mathcal{L}})\,. \tag{4.48}$$

We finally note from (4.44) and (4.41) that q generates a symmetry of the gauge fixed action:

$$\vec{q}\; S_{\mathcal{L}} = 0\,. \tag{4.49}$$

It is clear that q is the BRST generator in the gauge $\mathcal{L}$.

4.1.5 The coordinate description

We now sketch how the local description arises in the geometric formalism. Given a gauge $\mathcal{L}$, one can locally identify M and the total space of the bundle $N\mathcal{L}$. One chooses s-homogeneous coordinates z^α and z^*_α along $\mathcal{L}$ and the fiber of $N\mathcal{L}$ such that (z^*_α, z^α) are

Darboux coordinates for ω, i.e.:

$$\epsilon(z_\alpha^*) + \epsilon(z^\alpha) = \hat{1} \qquad \text{and} \qquad \omega_{\alpha^*\beta} = -\omega_{\beta\alpha^*} = \delta_{\alpha,\beta}\,. \tag{4.50}$$

and such that $s(z_\alpha^*) + s(z^\alpha) = -1$.[17]

In this case, the odd symplectic form reduces to:

$$\omega = dz_\alpha^* \wedge dz^\alpha\,. \tag{4.51}$$

One can identify z^α and z_α^* with the fields and antifields of the traditional formalism. In such coordinates, the BV bracket has the familiar form:

$$\{F, G\} = \frac{\partial_r F}{\partial z^\alpha}\frac{\partial_l G}{\partial z_\alpha^*} - \frac{\partial_r F}{\partial z_\alpha^*}\frac{\partial_l G}{\partial z^\alpha}\,, \tag{4.52}$$

and the lagrangian manifold $\mathcal{L}$ is locally described by the equations $z_\alpha^* = 0$.

4.1.6 The geometric meaning of BV 'quantization'

The procedure of BV 'quantization' translates as follows. One starts with a so-called 'classical gauge' $\mathcal{L}$, and with a *classical action* $S_{\mathcal{L}} = S_{BV}|_{\mathcal{L}}$ defined on $\mathcal{L}$. This gauge is typically not convenient for the purpose of quantization, in that $S_{\mathcal{L}}$ is degenerate (has degenerate hessian) on $\mathcal{L}$, which in field-theoretic applications leads to 'infinite factors' in the path integral and the impossibility of defining propagators. A degenerate hessian signals the existence of flat directions for S, which in conveniently chosen local coordinates $z = (z^a)$ means that S depends only on a subset $x = (z^j)$ of (z^a), which in practice is the subset of coordinates with ghost degree $s(z^j) = 0$. To avoid such problems, one picks another gauge $\mathcal{L}'$ such that the hessian of $S_{\mathcal{L}'}$ is nondegenerate. This allows for the definition of propagators in the new gauge, which provides a starting point for perturbative renormalization of the associated path integral.

The BV procedure can be seen as a systematic approach to performing the change of gauge from $\mathcal{L}$ to $\mathcal{L}'$. This is done by first extending the classical action $S_{\mathcal{L}}$ to the BV action S_{BV}, and then restricting the latter to $\mathcal{L}'$ to obtain a candidate for a meaningful definition of the path integral (this process can be described locally in the traditional language of gauge-fixing fermions). In its most general form [27], the central result is that two gauges $\mathcal{L}_1$ and $\mathcal{L}_2$ are physically equivalent provided that their bodies (even parts) are homologous in the body of M and that the BV action satisfies the quantum generalization of the classical master equation. It is important to realize, however, that this statement need not be valid in more general situations. There is no reason to expect that 'topologically inequivalent' gauges $\mathcal{L}_1$ and $\mathcal{L}_2$ lead to equivalent path integrals.

4.1.7 The BV algorithm

The full BV data is rarely known in practical applications. In a typical situation, one is only given the action $S := S_{\mathcal{L}}$ in a classical gauge. Since the classical gauge is degenerate,

[17]Note that one need not assume $\epsilon(z^\alpha) = s(z^\alpha) \pmod 2$ and $\epsilon(z_\alpha^*) = s(z_\alpha^*) \pmod 2$. In fact, this is impossible to arrange if the classical action S contains Grassmann-odd variables.

one has no apriori knowledge of any of the data M, ω or S_{BV}. In this case, one recovers a BV system $(M, \omega, S_{BV}, \mathcal{L})$ (such that $S_{BV}|_{\mathcal{L}} = S$) in the following constructive manner. First, one has to decide on a gauge algebra, i.e. an algebra of symmetries of S. It should be stressed that the choice of gauge algebra is not uniquely determined by S, since one can insist to choose a strict subalgebra of the maximal algebra of gauge symmetries of the classical action;[18] this choice is dictated by the physical interpretation of the model. Then one constructs a BV system through the following two-step procedure:

1. Perform the BRST extension of $S(x)$, by applying the BRST procedure for the given gauge algebra (the precise choice of algebra influences the result of this step). This enlarges the classical system from the set of classical fields x to the set of BV fields $z^\alpha = (x, c_1, c_2, \ldots)$, where c_k are ghosts at generation k. It also produces a nilpotent odd vector field q (the BRST generator) acting on the enlarged collection of fields. The number of ghost generations is dictated by the degree of reducibility of the gauge algebra. After extracting the complete set of ghosts c_k, introduce antifields $z^*_\alpha = (x^*, c^*_1, c^*_2, \ldots)$ for the classical fields and ghosts. The BV fields and antifields (z^α, z^*_α) are identified with Darboux coordinates of M. This allows one to locally recover both the supermanifold M and the odd symplectic form ω upon using relation (4.51). The lagrangian submanifold $\mathcal{L}$ which defines the classical gauge is recovered through the equation $z^*_\alpha = 0$. The fibers of the lagrangian complement N of $T\mathcal{L}$ are locally defined by the directions $z^\alpha = ct$.

   Knowledge of the correct collection of BV fields and antifields allows one to enlarge the degenerate classical action $S(x)$ by adding the so-called first order action $S_1(z^*, z) = z^*_\alpha q^\alpha(z)$. S and S_1 are the first two terms in the expansion of the BV action in antifields.

2. The odd symplectic form ω recovered in the first step defines the BV bracket $\{\cdot, \cdot\}$. To recover the full BV action, one must solve the master equation $\{S_{BV}, S_{BV}\} = 0$ with the ansatz $S_{BV} = \sum_{k \geq 0} S_k$, where S_k is the k^{th} order term in the Taylor expansion in antifields. The first two terms $S_0 = S$ and S_1 are known from step 1.

4.2 Realization of the geometric data and check of the master equation

4.2.1 The graded P-manifold

The supermanifold. Remember that the unextended total boundary space $\mathcal{H}$ is a $\mathbb{Z}$-graded vector space with respect to the worldsheet U(1) charge $|\cdot|$. The mod 2 reduction of $|\cdot|$ makes $\mathcal{H}$ into a vector superspace ($\mathbb{Z}_2$-graded vector space):

$$\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}} , \tag{4.53}$$

where:

$$\begin{aligned} \mathcal{H}_{\text{even}} &= \{u \in \mathcal{H} \mid |u| = \text{even}\} = \oplus_{k+n-m=\text{even}} \Omega^k(L) \otimes \Gamma(\text{Hom}(E_m, E_n)) , \\ \mathcal{H}_{\text{odd}} &= \{u \in \mathcal{H} \mid |u| = \text{odd}\} = \oplus_{k+n-m=\text{odd}} \Omega^k(L) \otimes \Gamma(\text{Hom}(E_m, E_n)) . \end{aligned} \tag{4.54}$$

[18] We shall encounter this phenomenon in section 7, when discussing the BV action for D-brane pairs with relative grading greater than one.

To construct the supermanifold relevant for the geometric BV formalism, we consider a new $\mathbb{Z}$-grading s on $\mathcal{H}$ which is related to $| \cdot |$ by:

$$s(u) = 1 - |u| \,. \tag{4.55}$$

The vector space $\mathcal{H}$ endowed with this grading will be denoted by $\tilde{\mathcal{H}}$. The mod 2 reduction of s makes $\tilde{\mathcal{H}}$ into a vector superspace:

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\rm even} \oplus \tilde{\mathcal{H}}_{\rm odd} \,, \tag{4.56}$$

where $\tilde{\mathcal{H}}_{\rm even} = \mathcal{H}_{\rm odd}$ and $\tilde{\mathcal{H}}_{\rm odd} = \mathcal{H}_{\rm even}$.

It is clear that the superspaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ differ by parity change. They define infinite-dimensional complex linear supermanifolds (in the sense of Berezin), which we denote by $L(\tilde{\mathcal{H}})$ and $L(\mathcal{H})$. Their G-valued points define De Witt-Rogers supermanifolds:

$$M := L(\tilde{\mathcal{H}})(G) = (\tilde{\mathcal{H}} \otimes G)^0 = \tilde{\mathcal{H}}^0_e = \mathcal{H}^1_e \,, \qquad P = L(\mathcal{H})(G) = (\mathcal{H} \otimes G)^0 = \mathcal{H}^0_e \,, \tag{4.57}$$

where we defined $\tilde{\mathcal{H}}_e := \tilde{\mathcal{H}} \otimes G$. In these equations, $\mathcal{H}_e$ and $\tilde{\mathcal{H}}_e$ are viewed as vector superspaces, and we used the obvious relation:

$$\tilde{\mathcal{H}}_e = \Pi \mathcal{H}_e \,. \tag{4.58}$$

The graded manifold structure. We will mainly be interested in the linear supermanifold M, on which we now introduce a structure of $\mathbb{Z}$-graded manifold. For this, consider a homogeneous basis e_a of $\mathcal{H}$, with $|e_a| := |a| \Rightarrow s(e_a) = s_a := 1 - |a|$. Then every element $\hat{\phi}$ of $\mathcal{H}^1_e$ has the expansion:

$$\hat{\phi} = \sum_a e_a \otimes \hat{\phi}^a \,, \tag{4.59}$$

with $\hat{\phi}^a \in G$ and $g(\hat{\phi}^a) = \deg \hat{\phi} - |e_a| \pmod 2 = 1 - |e_a| \pmod 2 = s_a \pmod 2$. This allows us to define maps $z^a : \mathcal{H}_e \to G$ through $z^a(\hat{\phi}) = \hat{\phi}^a$, which have parities $\epsilon_a = g_a = s_a \pmod 2$ as G-valued functions. They give (global) coordinates on the supermanifold M. Coordinates for M obtained in this manner will be called *homogeneous linear coordinates*.

The collection of homogeneous linear coordinates associated to all homogeneous bases (e_a) of $\mathcal{H}$ forms a distinguished atlas for our supermanifold. Such coordinates are endowed not only with a $\mathbb{Z}_2$-degree $\epsilon_a = \epsilon(z^a)$, but also with a $\mathbb{Z}$-valued degree $s(z^a) := s_a = s(e_a)$, such that $\epsilon_a = s_a \pmod 2$. As explained in section 4.2., this can be used to define a $\mathbb{Z}$-grading s on the sheaf $\mathcal{F}$ of G-valued functions, if one restricts the latter to consist of functions which are polynomial in coordinates. This grading on $\mathcal{F}$ will play the role of ghost grading in the BV formalism.

Vector fields as nonlinear operators. Since M is a linear supermanifold, vector fields on M can be viewed as maps of the form $X : \hat{\phi} \to (\hat{\phi}, \mathbf{X}(\hat{\phi})) \in M \times \tilde{\mathcal{H}}_e$, where $\mathbf{X}$ is a (generally nonlinear) operator from M to $\tilde{\mathcal{H}}_e$. Even ($\epsilon_X = \hat{0}$) and odd ($\epsilon_X = \hat{1}$) vector fields correspond to operators $\mathbf{X}$ from M to $\mathcal{H}^1_e$ and M to $\mathcal{H}^0_e$ respectively.

In homogeneous linear coordinates z^a, the vector fields ∂^r_a correspond to the constant operators $\hat{\phi} \to e_a \otimes 1_G$. One has $s(\partial^r_a) = -s_a$ and $\epsilon(\partial^r_a) = \epsilon_a = s_a \pmod 2$. For an arbitrary vector field $X = \partial_a^r X_r^a$, one has $\mathbf{X}(\hat{\phi}) = e_a \otimes X_r^a(\hat{\phi})$ (with $X_r^a(\hat{\phi}) \in G$). Since $z^a(\mathbf{X}(\hat{\phi})) = X_r^a(\hat{\phi})$, the vector field X can be recovered from the operator $\mathbf{X}$ through the relation:

$$(z^a \overleftarrow{X})(\hat{\phi}) = X^a(\hat{\phi}) = z^a(\mathbf{X}(\hat{\phi}))\,. \tag{4.60}$$

Note that one can expand:

$$\mathbf{X}(\hat{\phi}) = \mathbf{X}_0 + \mathbf{X}_1(\hat{\phi}) + \mathbf{X}_2(\hat{\phi},\hat{\phi}) + \mathbf{X}_3(\hat{\phi},\hat{\phi},\hat{\phi}) + \cdots\,, \tag{4.61}$$

where $\mathbf{X}_k : (\tilde{\mathcal{H}} \otimes G)^{\otimes k} \to \tilde{\mathcal{H}} \otimes G$ are G-multilinear operators. It is not hard to check that the vector field X is s-homogeneous of degree σ if and only if $\mathbf{X}_k$ are s-homogeneous of degree $-\sigma$, i.e.:

$$s(\mathbf{X}_k(\hat{\phi}_1, \ldots, \hat{\phi}_k)) = s(\hat{\phi}_1) + \cdots + s(\hat{\phi}_k) - \sigma\,. \tag{4.62}$$

By localizing the description of vector fields to a point $\hat{\phi}$ of M, one obtains the identification:

$$T_{\hat{\phi}}M = \tilde{\mathcal{H}} \otimes G = \tilde{\mathcal{H}}_e\,, \tag{4.63}$$

which holds both as an isomorphism of right G-supermodules and as an isomorphism of $\mathbb{Z}$-graded vector spaces (the $\mathbb{Z}$-grading on both sides being given by the ghost degree s). It follows that the total space of the tangent bundle to M can be identified with:

$$TM = M \times \tilde{\mathcal{H}}_e\,. \tag{4.64}$$

The left G-supermodule structure on $T_{\hat{\phi}}M$ is defined through:[19]

$$\alpha X_{\hat{\phi}} = (-1)^{\epsilon_\alpha \epsilon(X_{\hat{\phi}})} X_{\hat{\phi}} \alpha\,, \tag{4.65}$$

where $\alpha \in G$ and $\epsilon_\alpha = g(\alpha)$.

The odd symplectic form. The extended bilinear form $\langle\cdot,\cdot\rangle_e$ on $\mathcal{H}_e$ allows us to define the following two-form on M:

$$\omega_{\hat{\phi}}(X_{\hat{\phi}}, Y_{\hat{\phi}}) := (-1)^{\epsilon(X_{\hat{\phi}})} \langle X_{\hat{\phi}}, Y_{\hat{\phi}}\rangle_e\,, \tag{4.66}$$

where $\hat{\phi}$ is a point in M and $X_{\hat{\phi}}, Y_{\hat{\phi}} \in T_{\hat{\phi}}M = \tilde{\mathcal{H}} \otimes G$ are tangent vectors to M at $\hat{\phi}$. It is easy to check that ω is an *odd symplectic form* on M, and that $s_\omega = -1$. The last statement follows upon choosing a homogeneous basis s_a of $\mathcal{H}$ and considering the coefficients of ω in this basis:

$$\omega_{ab} = (-1)^{\epsilon_a} \langle e_a, e_b\rangle = ct\,, \tag{4.67}$$

where we used the identification $\partial_a^r = e_a \otimes 1_G$ and the fact that $\langle e_a \otimes 1_G, e_b \otimes 1_G\rangle_e = \langle e_a, e_b\rangle$. The selection rules (3.22) allow us to restrict to the case $|e_a| + |e_b| = 3 \Leftrightarrow s_a + s_b = -1$. In this case, one has $s(\omega_{ab}) = -s_a - s_b - 1$, where we used the fact that ω_{ab} is a constant and thus $s(\omega_{ab}) = 0$. This implies that $s_\omega = -1$. We conclude that (M, ω) is a (De Witt-Rogers)

[19] Note that the exponent in (4.65) involves $\epsilon(X_{\hat{\phi}}) = \deg X_{\hat{\phi}} + \hat{1}$ and *not* $\deg X_{\hat{\phi}}$. This is due to the presence of parity reversal in the isomorphism of superspaces $T_{\hat{\phi}}M = \tilde{\mathcal{H}}_e = \Pi\mathcal{H}_e$.

graded P-manifold. Note that the restriction of ω to the even component $TM^0 \approx \mathcal{H}^1_e$ of the tangent bundle coincides with the form $\omega_0 = \langle \cdot, \cdot \rangle_e|_{M\times M}$ of equation (3.39). In fact, ω is completely determined by this restriction and by the requirement that it must be an odd form. This observation will be useful in section 7.

The extended action (3.34) is an (even) function defined on M. To check that its ghost degree equals zero, we first notice that the operator d and extended boundary product $*$ satisfy $s(d\hat{\phi}) = s(\hat{\phi}) - 1$ and $s(\hat{\phi}_1 * \hat{\phi}_2) = s(\hat{\phi}_1) + s(\hat{\phi}_2) - 1$. Since d and $*$ are right G-linear and bilinear respectively, they define a nonlinear operator $\mathbf{W}(\hat{\phi}) = \frac{1}{2}d\hat{\phi} + \frac{1}{3}\hat{\phi} * \hat{\phi}$ which is s-homogeneous of degree -1. This in turn defines a vector field W of ghost degree $+1$ on M. On the other hand, the identity operator $\mathbf{I} : \hat{\phi} \to \hat{\phi}$ defines a vector field I of ghost degree zero. It is then clear that S_e can be written in the purely geometric form:

$$S_e = \omega(I, W)\,, \tag{4.68}$$

which obviously has ghost degree zero (since ω has ghost degree -1). We are now ready to apply the geometric formalism to the system (M, S_e, ω) in order to show that the extended action satisfies the classical master equation.

4.2.2 The odd vector field Q

As discussed above, the odd hamiltonian vector field $Q := Q_{S_e}$ can be viewed as a (non-linear) map $\mathbf{Q}$ from M to P. To obtain an explicit formula for $\mathbf{Q}$, we compute the variation of $S_e(\hat{\phi})$ under an infinitesimal change of $\hat{\phi}$:

$$\delta S_e(\hat{\phi}) = \left\langle d\hat{\phi} + \frac{1}{2}[\hat{\phi}, \hat{\phi}]_*, \delta\hat{\phi} \right\rangle_e = -\omega_{\hat{\phi}}\left(d\hat{\phi} + \frac{1}{2}[\hat{\phi}, \hat{\phi}]_*, \delta\hat{\phi}\right), \tag{4.69}$$

which means that the differential of S_e has the form:

$$dS_e(X) = \omega(Q, X)\,, \tag{4.70}$$

with:

$$\mathbf{Q}(\hat{\phi}) = -\left(d\hat{\phi} + \frac{1}{2}[\hat{\phi}, \hat{\phi}]_*\right). \tag{4.71}$$

4.2.3 Check of the master equation

Relation (4.71) implies:

$$\omega(Q, Q)(\hat{\phi}) = \omega_{\hat{\phi}}(\mathbf{Q}(\hat{\phi}), \mathbf{Q}(\hat{\phi})) = -\langle d\hat{\phi}, d\hat{\phi}\rangle_e - 2\langle d\hat{\phi}, \hat{\phi}^{*2}\rangle_e - \langle \hat{\phi}^{*2}, \hat{\phi}^{*2}\rangle_e\,, \tag{4.72}$$

where we use the notation ϕ^{*n} to indicate the n^{th} power of ϕ computed with the product $*$. It is easy to check that all three terms vanish upon using the properties of ω and the condition $\deg \hat{\phi} = \hat{1}$. Indeed:

$$\langle d\hat{\phi}, d\hat{\phi}\rangle_e = \langle \hat{\phi}, d^2\hat{\phi}\rangle_e = 0\,, \tag{4.73}$$

$$\begin{aligned}\langle d\hat{\phi}, \hat{\phi}^{*2}\rangle_e &= \langle \hat{\phi}, (d\hat{\phi}) * \hat{\phi}\rangle_e - \langle \hat{\phi}, \hat{\phi} * d\hat{\phi}\rangle_e = -\langle (d\hat{\phi}) * \hat{\phi}, \hat{\phi}\rangle_e - \langle \hat{\phi}^{*2}, d\hat{\phi}\rangle_e \\ &= -2\langle d\hat{\phi}, \hat{\phi}^{*2}\rangle_e \Longrightarrow \langle d\hat{\phi}, \hat{\phi}^{*2}\rangle_e = 0\,,\end{aligned} \tag{4.74}$$

and finally:

$$\langle\hat{\phi}^{*2},\hat{\phi}^{*2}\rangle_e=\langle\hat{\phi},\hat{\phi}^{*3}\rangle_e=-\langle\hat{\phi}^{*3},\hat{\phi}\rangle_e=-\langle\hat{\phi}^{*2},\hat{\phi}^{*2}\rangle_e\implies\langle\hat{\phi}^{*2},\hat{\phi}^{*2}\rangle_e=0\,. \tag{4.75}$$

We conclude that $\omega(\mathbf{Q},\mathbf{Q})=0$, which in view of equations (4.41) implies that S_e satisfies the classical master equation $\{S_e,S_e\}=0$, with respect to the BV bracket induced by ω.

4.2.4 The classical gauge

Let us consider the decompositions:

$$M=\oplus_s M_s\,,\qquad P=\oplus_s P_s\,, \tag{4.76}$$

where $M_s=\tilde{\mathcal{H}}^s\otimes G_{s\,(\mathrm{mod}\,2)}=\mathcal{H}^{1-s}\otimes G_{s\,(\mathrm{mod}\,2)}$, $P_s=\tilde{\mathcal{H}}^s\otimes G_{(1-s)\,(\mathrm{mod}\,2)}=\mathcal{H}^{1-s}\otimes G_{(1-s)\,(\mathrm{mod}\,2)}$ denote the the subspaces of M and P spanned by elements $\hat{\phi}$ of ghost number s. The space $M_0=\tilde{\mathcal{H}}^0\otimes G_0=\mathcal{H}^1\otimes G_0$ was considered in equation (3.37) of section 3.2. As mentioned there, the unextended string field ϕ can be related to the component $\hat{\phi}_0$ of $\hat{\phi}$ along this subspace:

$$\hat{\phi}_0=\phi\otimes 1_G+\hat{\phi}_0^\mu\xi_\mu\,, \tag{4.77}$$

where ξ^μ are the odd generators of G (see relation 3.1). Since it is easy to include the evaluation map ev_G in relations such as (3.38), we shall denote $\hat{\phi}_0$ by ϕ in order to simplify notation. This allows us to view the unextended string field as the component of $\hat{\phi}$ along M_0.

The selection rule (3.21) for the symplectic form implies that the subspaces:

$$\begin{aligned}\mathcal{L}&=\{\hat{\phi}\in M|s(\hat{\phi})\geq 0\}=\oplus_{s\geq 0}M_s\\ \mathcal{N}&=\{\hat{\phi}\in M|s(\hat{\phi})<0\}=\oplus_{s<0}M_s\end{aligned} \tag{4.78}$$

give a lagrangian decomposition of (M,ω) (viewed as an odd symplectic vector space). In particular, $\mathcal{L}$ is a lagrangian submanifold of M, and we can identify $T\mathcal{L}=\mathcal{L}\times\mathcal{L}$ and $N\mathcal{L}=\mathcal{L}\times\mathcal{N}$ (then $TM|_{\mathcal{L}}(s)=\mathcal{L}\times M_s$).

The same selection rule shows that the restriction of S_e to $\mathcal{L}$ coincides with S (up to application of ev_G). It follows that S_e plays the role of BV action for the classical action S, while $\mathcal{L}$ describes the associated 'classical gauge'. Since $S_e|_{\mathcal{L}}=S$ depends only on the component of $\hat{\phi}$ along M_0, the classical action $S_{\mathcal{L}}$ has degenerate hessian on $\mathcal{L}$. Note that the classical gauge $\mathcal{L}$ is entirely determined by the ghost grading s.

Following standard BV procedure, we define BV *fields* $\boldsymbol{\phi}\in\mathcal{L}$ and *antifields* $\boldsymbol{\phi}^*\in\mathcal{N}$ as the components of $\hat{\phi}$ along $\mathcal{L}$ and $\mathcal{N}$. This gives the decomposition $\hat{\phi}=\boldsymbol{\phi}+\boldsymbol{\phi}^*$. The component $\phi\in M_0$ of $\boldsymbol{\phi}$ is the unextended string field, while the higher components $c_s\in M_s$ ($s\geq 1$) play the role of ghosts. The highest component $\phi^*\in M_{-1}$ of $\boldsymbol{\phi}^*$ is the antifield of ϕ, while the lower components $c_s^*\in M_{-1-s}$ ($s\geq 1$) are the antifields of c_s. Hence one has the decomposition:

$$\hat{\phi}=\cdots+c_2^*+c_1^*+\phi^*+\phi+c_1+c_2+\cdots\,, \tag{4.79}$$

which we also write in the form:

$$\hat{\phi}=\oplus_s\phi_s\,, \tag{4.80}$$

where $\phi_0 := \phi$, $\phi_{-1} = \phi^*$, $\phi_s = c_s$ and $\phi_{-1-s} = c_s^*$ for $s \geq 1$. We have $\boldsymbol{\phi} = \oplus_{s\geq 0}\phi_s$ and $\boldsymbol{\phi}^* = \oplus_{s<0}\phi_s = \oplus_{s\geq 0}\phi_s^*$. Note the relations:

$$\phi_{-1-s} = \phi_s^* \quad \text{for } s \geq 0\,, \quad rk\phi_s^* + rk\phi_s = 3\,, \quad \Delta(\phi_s^*) + \Delta(\phi_s) = 0\,, \quad g(\phi_s^*) + g(\phi_s) = \hat{1}\,, \tag{4.81}$$

which are due to the selection rules for ω.

4.2.5 BRST transformations in the classical gauge

Applying parity change to the bundle decomposition $TM|_{\mathcal{L}} = T\mathcal{L} \oplus T\mathcal{N}$ gives $\Pi TM|_{\mathcal{L}} = \Pi T\mathcal{L} \oplus \Pi N\mathcal{L}$, with $\Pi T\mathcal{L} = \mathcal{L} \times \Pi\mathcal{L}$ and $\Pi N\mathcal{L} = \mathcal{L} \times \Pi\mathcal{N}$, with $\Pi\mathcal{L}$ and $\Pi\mathcal{N}$ given by the following complementary subspaces of $\mathcal{H}_e$:

$$\Pi\mathcal{L} = \oplus_{s\geq 0}P_s\,, \qquad \Pi\mathcal{N} = \oplus_{s<0}P_s\,, \tag{4.82}$$

where used the obvious identity $\Pi M_s = P_s$. Since $\mathcal{L}$ is a vector space and the bundle $\Pi T\mathcal{L} = \mathcal{L} \times \Pi\mathcal{L}$ is trivial, the odd vector field q of (4.44) can be identified with an even nonlinear operator $\mathbf{q}$ from $\mathcal{L}$ to $\Pi\mathcal{L}$. Relation (4.44) then translates as:

$$\mathbf{q}(\boldsymbol{\phi}) = T(\mathbf{Q}(\boldsymbol{\phi})) = -T\left(d\boldsymbol{\phi} + \frac{1}{2}[\boldsymbol{\phi}, \boldsymbol{\phi}]_*\right)\,, \tag{4.83}$$

where T is the projector of P onto $\Pi\mathcal{L}$, taken parallel with the subspace $\Pi\mathcal{N}$ (the projector on parity changed BV fields).

4.2.6 Expansion of S_e in antifields

We end this section by writing the extended action in a form which will be useful later. It is easy to check that:

$$S_e(\hat{\phi}) = S(\phi) - \langle \boldsymbol{\phi}^*, \mathbf{Q}(\boldsymbol{\phi})\rangle_e + \langle \boldsymbol{\phi}, \boldsymbol{\phi}^* \ * \boldsymbol{\phi}^*\rangle_e = S(\phi) - \langle \boldsymbol{\phi}^*, \mathbf{q}(\boldsymbol{\phi})\rangle_e + \langle \boldsymbol{\phi}, \boldsymbol{\phi}^* \ * \boldsymbol{\phi}^*\rangle_e\,. \tag{4.84}$$

This expression corresponds to the expansion of S_e in antifields around the classical gauge $\mathcal{L}$.

5. Graded D-brane pairs

5.1 'Component' description of graded pairs

If one restricts to the case of graded D-brane pairs, our string field theory can be described as follows (figure 1). Using the labels a, b to denote the associated graded branes, with underlying flat bundles E_a and E_b living on L, one can take the grade of a to be zero without loss of generality. If n denotes the grade of b, then the relative grade $\mathrm{grade}(b) - \mathrm{grade}(a)$ is n. The theory has four boundary sectors, which we denote by $\mathrm{Hom}(a,a)$, $\mathrm{Hom}(b,b)$ (the diagonal sectors) and $\mathrm{Hom}(a,b)$, $\mathrm{Hom}(b,a)$ (the off-diagonal, or boundary condition changing sectors). They are the off-shell spaces of states for strings stretching from a to

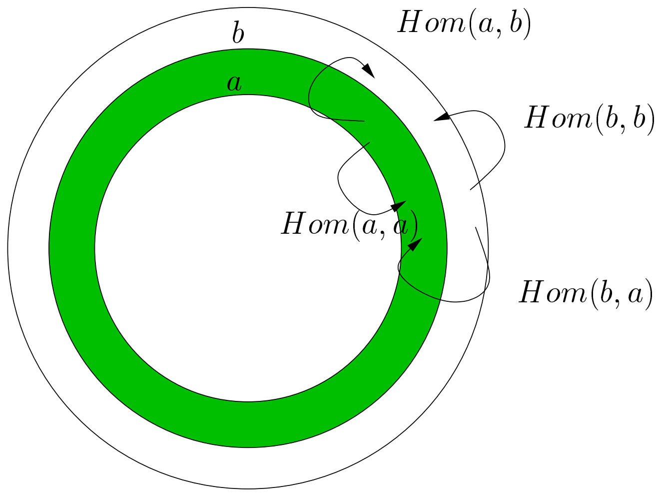

Figure 1: Boundary sectors for a pair of graded D-branes wrapping the same special lagrangian cycle. The two D-branes a and b are thickened out for clarity, though their (classical) thickness is zero.

a, b to b, a to b and b to a respectively. The localization arguments of [17], combined with the shift of U(1) charge discussed in [1], lead to the identifications:

$$\begin{aligned} \mathrm{Hom}^k(a,a) &= \Omega^k(\mathrm{End}(E_a)) \,, & \mathrm{Hom}^k(b,b) &= \Omega^k(\mathrm{End}(E_b)) \\ \mathrm{Hom}^k(a,b) &= \Omega^{k-n}(\mathrm{Hom}(E_a,E_b)) \,, & \mathrm{Hom}^k(b,a) &= \Omega^{k+n}(\mathrm{Hom}(E_b,E_a)) \,, \end{aligned} \tag{5.1}$$

where k is the worldsheet U(1) charge. The relation between the charge $|\cdot|$ and form rank is given by (2.31):

$$|u_{AB}| = rku_{AB} + \mathrm{grade}(B) - \mathrm{grade}(A) \qquad \text{for } u_{AB} \in \mathrm{Hom}(A,B) \,, \tag{5.2}$$

for all $A, B \in \{a, b\}$. In equations (5.1), we let k take any integer value and define the space of forms of negative ranks (or ranks greater than three) on L to be zero. Thus non-vanishing states in the sectors $\mathrm{Hom}(a,b)$ and $\mathrm{Hom}(b,a)$ have charges $k = n, n+1, n+2$ or $n+3$ and $k = -n, -n+1, -n+2, -n+3$ respectively. It follows that the total boundary space $\mathcal{H} = \mathrm{Hom}(a,a) \oplus \mathrm{Hom}(b,a) \oplus \mathrm{Hom}(a,b) \oplus \mathrm{Hom}(b,b)$ has non-vanishing components of charges $k = \{-n, \ldots, -n+3\} \cup \{n, \ldots, n+3\} \cup \{0, \ldots, 3\}$.

The various boundary sectors carry BRST operators d_{aa}, d_{ab}, d_{ba} and d_{bb} given by the covariant differentials twisted with the flat connections A_a and A_b. One also has a modification of the boundary product:

$$u_{BC} \bullet u_{AB} = (-1)^{[\mathrm{grade}(B) - \mathrm{grade}(C)]\, rku_{AB}}\, u_{BC} \wedge u_{AB} \,, \tag{5.3}$$

for all $A, B, C \in \{a, b\}$.

Upon using this data, one can write the explicit form of the string field action (2.25) for such a system. In this case, the graded bundle is $\mathbf{E} = E_a \oplus E_b$, and the string field ϕ is a (worldsheet charge one) element of the total boundary space $\mathcal{H} = \Omega^*(L) \otimes \Gamma(\mathrm{End}(\mathbf{E}))$.

It can be represented as a matrix of bundle-valued forms:

$$\phi = \begin{bmatrix} \phi_{aa}^{(1)} & \phi_{ba}^{(1+n)} \\ \phi_{ab}^{(1-n)} & \phi_{bb}^{(1)} \end{bmatrix}, \tag{5.4}$$

with:

$$\begin{aligned} \phi_{aa}^{(1)} &\in \mathrm{Hom}^1(a,a) = \Omega^1(\mathrm{End}(E_a)), & \phi_{bb}^{(1)} &\in \mathrm{Hom}^1(b,b) = \Omega^1(\mathrm{End}(E_b)) \\ \phi_{ab}^{(1-n)} &\in \mathrm{Hom}^1(a,b) = \Omega^{1-n}(\mathrm{Hom}(E_a,E_b)), & \phi_{ba}^{(1+n)} &\in \mathrm{Hom}^1(b,a) = \Omega^{1+n}(\mathrm{Hom}(E_b,E_a)), \end{aligned} \tag{5.5}$$

where the superscripts in round brackets indicate form rank.

Given two states $u, v \in \mathcal{H}$, represented by the matrices $u = \begin{bmatrix} u_{aa} & u_{ba} \\ u_{ab} & u_{bb} \end{bmatrix}$ and $v = \begin{bmatrix} v_{aa} & v_{ba} \\ v_{ab} & v_{bb} \end{bmatrix}$, their boundary product (2.6) is given by:

$$u \bullet v := \begin{bmatrix} u_{aa} \bullet v_{aa} + u_{ba} \bullet v_{ab} & u_{aa} \bullet v_{ba} + u_{ba} \bullet v_{bb} \\ u_{ab} \bullet v_{aa} + u_{bb} \bullet v_{ab} & u_{bb} \bullet v_{bb} + u_{ab} \bullet v_{ba} \end{bmatrix}. \tag{5.6}$$

We also have the fiberwise supertrace on $\mathrm{End}(\mathbf{E})$:

$$str \begin{bmatrix} u_{aa} & u_{ba} \\ u_{ab} & u_{bb} \end{bmatrix} = \mathrm{tr}_a(u_{aa}) + (-1)^n\, \mathrm{tr}_b(u_{bb}), \tag{5.7}$$

and the total worldsheet BRST operator:

$$d \begin{bmatrix} u_{aa} & u_{ba} \\ u_{ab} & u_{bb} \end{bmatrix} = \begin{bmatrix} d_{aa} u_{aa} & d_{ba} u_{ba} \\ d_{ab} u_{ab} & d_{bb} u_{bb} \end{bmatrix}, \tag{5.8}$$

as well as the boundary bilinear form:

$$\langle u, v\rangle = \int_L \mathrm{str}(u \bullet v) = \int_L [\mathrm{tr}_a(u_{aa} \bullet v_{aa} + u_{ba} \bullet v_{ab}) + (-1)^n\, \mathrm{tr}_b(u_{ab} \bullet v_{ba} + u_{bb} \bullet v_{bb})\ ], \tag{5.9}$$

where tr_a and tr_b denote the fiberwise trace on the bundles $\mathrm{End}(E_a)$ and $\mathrm{End}(E_b)$.

With these notations, the string field action (2.25) expands as:

$$\begin{aligned} S(\phi) &= \int_L \mathrm{str}\left[\frac{1}{2}\phi \bullet d\phi + \frac{1}{3}\phi \bullet \phi \bullet \phi\right] \\ &= \frac{1}{2}\int_L [\mathrm{tr}_a(\phi_{aa} \bullet d\phi_{aa} + \phi_{ba} \bullet d\phi_{ab}) + (-1)^n\, \mathrm{tr}_b(\phi_{bb} \bullet d\phi_{bb} + \phi_{ab} \bullet d\phi_{ba})] + \\ &\quad + \frac{1}{3}\int_L [\mathrm{tr}_a(\phi_{aa} \bullet \phi_{aa} \bullet \phi_{aa} + \phi_{aa} \bullet \phi_{ba} \bullet \phi_{ab} + \phi_{ba} \bullet \phi_{bb} \bullet \phi_{ab} + \phi_{ba} \bullet \phi_{ab} \bullet \phi_{aa})] + \\ &\quad + \frac{1}{3}\int_L (-1)^n [\mathrm{tr}_b(\phi_{bb} \bullet \phi_{bb} \bullet \phi_{bb} + \phi_{bb} \bullet \phi_{ab} \bullet \phi_{ba} + \phi_{ab} \bullet \phi_{aa} \bullet \phi_{ba} + \phi_{ab} \bullet \phi_{ba} \bullet \phi_{bb})]\,. \end{aligned} \tag{5.10}$$

Since the rank of a form on the three-cycle L lies between 0 and 3, one can distinguish the cases $n = -2, -1, 0, 1, 2$ as well as the 'diagonal' case $|n| \geq 3$. Notice that one can easily translate from negative to positive n by reversing the roles of a and b, so we can further restrict to $n = 0, 1$ or 2. The case $n = 0$ gives the usual Chern-Simons theory on the direct sum bundle $E_a \oplus E_b$. The case $n \geq 3$ gives a graded sum of two Chern-Simons theories. Hence the interesting cases are $n = 1$ and $n = 2$.

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5.2 Comparison of worldsheet BRST cohomologies

Let us compare the physical (charge one) worldsheet BRST cohomology for the cases $n = 0, 1, 2$. The cohomology of the BRST operator d has the direct sum decomposition:

$$H^1_d(\mathcal{H}) = H^1(\mathrm{End}(E_a)) \oplus H^1(\mathrm{End}(E_b)) \oplus H^{1+n}(\mathrm{Hom}(E_b, E_a)) \oplus H^{1-n}(\mathrm{Hom}(E_a, E_b))\,. \tag{5.11}$$

Hence it suffices to compare the 'off-diagonal components' $H^1_{od} = H^{1+n}(\mathrm{Hom}(E_b, E_a)) \oplus H^{1-n}(\mathrm{Hom}(E_a, E_b))$. Since $H^{1+n}(\mathrm{Hom}(E_b, E_a)) \approx H^{2-n}(\mathrm{Hom}(E_a, E_b))$ by Poincaré duality, one obtains:

1. $(n = 0)$ $H^1_{od} \approx H^1(\mathrm{Hom}(E_a, E_b)) \oplus H^2(\mathrm{Hom}(E_a, E_b))$.

2. $(n = 1)$ $H^1_{od} \approx H^0(\mathrm{Hom}(E_a, E_b)) \oplus H^1(\mathrm{Hom}(E_a, E_b))$.

3. $(n = 2)$ $H^1_{od} \approx H^0(\mathrm{Hom}(E_a, E_b))$.

4. $(n \geq 3)$ $H^1_{od} \approx 0$.

It is clear from this that $H^1_d(\mathcal{H})$ will generally be different for various n. For the simple case when $E_a \approx E_b = \mathcal{O}_L$ (the trivial line bundle on L), endowed with the trivial flat structures, one obtains:

1. $(n = 0)$ $\dim H^1_{od} = 2b_1(L)$.

2. $(n = 1)$ $\dim H^1_{od} = 1 + b_1(L)$.

3. $(n = 2)$ $\dim H^1_{od} = 1$.

4. $(n \geq 3)$ $\dim H^1_{od} = 0\,,$

where $b_1(L)$ is the first Betti number of L (we assume that L is connected, so that $\dim H^0(L) = 1$). Since L is compact, the dimension $\dim H^1_d(\mathcal{H}) = 2b_1(L) + \dim H^1_{od}$ counts the number of independent physical degrees of freedom. It follows that our theories are generally inequivalent. The case $b_1(L) \geq 2$ (for example a special lagrangian 3-torus, with $b_1(L) = 3$) allows us to distinguish between all four classes based on this simple argument. Including the 'conjugate' cases $n < 0$, we conclude that there are in general *six* distinct types of D-brane pairs, which shows that the $\mathbb{Z}$-valued D-brane grade has physical consequences. This is true even though D-brane pairs whose relative gradings coincide modulo two (such as the pairs with $n = -1$, $n = +1$, or the pairs $n = -2$, $n = 0$ and $n = 2$) have the same classical master action. As mentioned above, the difference between such theories can be understood at the BV level as resulting from inequivalent choices for the ghost grading and classical gauge.

6. Composite formation and acyclicity

6.1 Vacuum shifts and D-brane composites

As discussed in some detail in [6] (upon following the general framework of [2, 3]), D-brane composite formation can be described by the simple device of shifting the string vacuum. This results from the observation [2] that a vacuum shift will generally break the decomposition of the total boundary space $\mathcal{H}$ into boundary sectors, thereby forcing a change in our D-brane interpretation.[20] In the case of graded D-brane pairs, this process can be described as follows. Suppose that we are given a (worldsheet charge one) solution ϕ to the string field equations of motion:

$$\frac{\delta S}{\delta \phi} = 0 \Longleftrightarrow d\phi + \phi \bullet \phi = 0\,, \tag{6.1}$$

and with the property that at least one of the components $\phi_{ab}^{(1-n)}$ and $\phi_{ba}^{(1+n)}$ is nonzero. If one shifts the string vacuum through ϕ, then the BRST operator around the new vacuum is given by:

$$d'_\phi u = du + [\phi, u]_\bullet = du + \phi \bullet u - (-1)^{|u|} u \bullet \phi\,, \tag{6.2}$$

where $[u,v]_\bullet = u \bullet v - (-1)^{|u||v|} v \bullet u$ stands for the *graded* commutator with respect to the boundary product $\bullet$ and U(1) charge $|\cdot|$, and u is an element of the total boundary space $\mathcal{H}$. The string field equations of motion (6.1) are equivalent with the condition that d'_ϕ squares to zero. The important observation is that d'_ϕ does not preserve the original boundary sectors $\mathrm{Hom}(a,a)$, $\mathrm{Hom}(a,b)$ etc, due to non-vanishing of either $\phi_{ab}^{(1-n)}$ or $\phi_{ba}^{(1+n)}$. Hence one cannot interpret the new background as a collection of two D-branes. In fact, there is no decomposition of $\mathcal{H}$ into new boundary sectors which would satisfy the basic conditions for such an interpretation (these conditions can be formulated [2] as the existence of a category structure compatibile with d'_ϕ and with the boundary product and bilinear form). Hence the new vacuum must be interpreted as a single object, namely as a composite of the original branes. In this case, $\mathcal{H}$ is viewed as the boundary sector of strings 'stretching from this composite to itself', though whether such an interpretation can be implemented in some sort of sigma model requires a case by case analysis. The issue of some sigma model representation for such composites is secondary for our purpose, since we are interested in *off-shell* string dynamics, for which it is natural to take the string field theory approach as fundamental. From our perspective, *all* D-brane composites constructed in this manner are equally 'fundamental', and their inclusion in the theory is a dynamical requirement. Whether the entirety of this dynamics admits some open sigma model interpretation is irrelevant for our considerations.

6.2 Graded D-brane composites as a representation of the extended moduli space

Let us consider an ungraded topological D-brane a wrapping L, i.e. a flat bundle E on L, with underlying connection A. The string field theory for strings whose endpoints lie on a

[20] This process admits an elegant mathematical description in the language of differential graded categories [2], but no familiarity with category theory is required for reading the present paper.

is described [17] by the Chern-Simons field theory on E. The associated moduli space $\mathcal{M}$ of string vacua is the moduli space of flat connections on E, which can be described through rank one solutions $\phi \in \Omega^1(\mathrm{End}(E))$ to the Maurer-Cartan equation for the commutator algebra of the graded associative algebra $(\Omega^*(L), \wedge)$, which is the boundary algebra of topological A-type strings stretching from a to a:

$$d\phi + \frac{1}{2}[\phi, \phi]_\wedge = 0 \Longleftrightarrow d\phi + \phi \wedge \phi = 0\,, \tag{6.3}$$

divided through the gauge group generated by transformations of the form:

$$\phi \to \phi - d\alpha - [\phi, \alpha]_\wedge\,. \tag{6.4}$$

In these equations, $[u, v]_\wedge = u \wedge v - (-1)^{rku\; rkv} v \wedge u$ is the graded commutator built from the usual wedge product of $\mathrm{End}(E)$-valued forms and taken with respect to the grading given by form rank. Since $rk\alpha = rk\phi - 1$ it follows that in equation (6.4) the graded commutator is a genuine commutator. The tangent space to this moduli space is given by $H^1(\mathrm{End}(E))$, as can be seen by considering the linearized form of (6.5) and (6.4).

An important problem in open topological string theory and mirror symmetry is to understand the significance of the so-called *extended moduli space* $\mathcal{M}_e$, obtained by considering deformations along directions in the other cohomology groups, namely $H^0(\mathrm{End}(E))$, $H^2(\mathrm{End}(E))$ and $H^3(\mathrm{End}(E))$. The extended moduli space can be defined formally through the so-called 'extended deformation theory' of [41], which gives a precise technique for constructing a (formal) supermanifold $\mathcal{M}_e$ whose tangent space at the origin is given by the full cohomology group $H^*(\mathrm{End}(E))$. Since $H^1(\mathrm{End}(E))$ is a subspace of $H^*(\mathrm{End}(E))$, the unextended moduli space $\mathcal{M}$ can be identified with a submanifold of $\mathcal{M}_e$. Points in the complement $\mathcal{M}_e - \mathcal{M}$ are understood as 'generalized' vacua of the open string theory in the given boundary sector, i.e. topological string backgrounds which do not have an immediate geometric interpretation.

The formal approach of extended deformation theory tells us nothing about the physical significance of the generalized backgrounds described by points lying in $\mathcal{M}_e - \mathcal{M}$. This problem has obstructed efforts to fulfill the program outlined in [42, 43] of gaining a better understanding of mirror symmetry through a study of extended moduli spaces. It is a remarkable fact (pointed out in [6]) that the theory of graded D-branes allows us to give an *explicit* physical description of such points.

To understand this, let us consider the case of graded D-branes a and $b = a[n]$ on L, whose underlying flat bundles coincide ($E_a = E_b := E$) and whose grades differ by n. In this case, the string field ϕ belongs to the space $\mathcal{H}^1 = \Omega^1(\mathrm{End}(E))^{\oplus 2} \oplus \Omega^{1-n}(\mathrm{End}(E)) \oplus \Omega^{1+n}(\mathrm{End}(E))$, and the classical equation of motion (6.1) is replaced by the Maurer-Cartan equation for the commutator algebra of the boundary algebra $\mathcal{H}$, which gives the equation of motion (6.1) for the string field theory (5.10):

$$d\phi + \frac{1}{2}[\phi, \phi]_\bullet = 0\,, \tag{6.5}$$

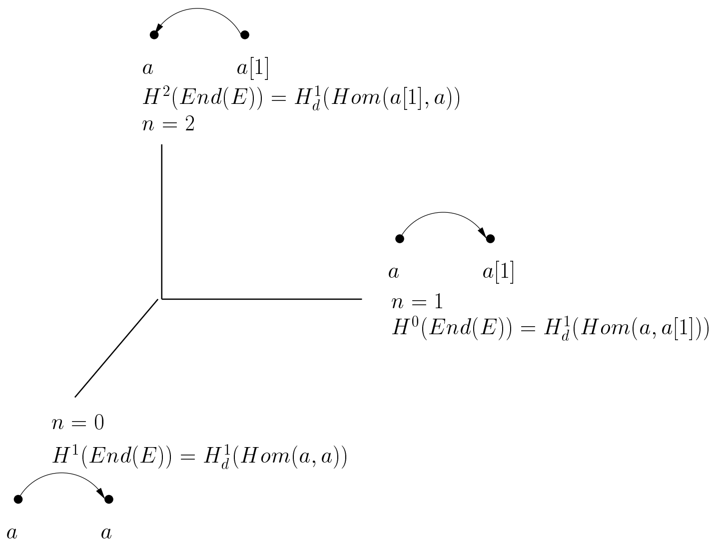

Figure 2: Points in the extended moduli space of an ungraded D-brane a can be represented through condensation of boundary condition changing operators between a and one of its higher shifts $b = a[n]$. For example, deformations along $H^0(\mathrm{End}(E))$, $H^1(\mathrm{End}(E))$ and $H^2(\mathrm{End}(E))$ can be represented by condensing operators in the sectors $\mathrm{Hom}(a, a[1])$, $\mathrm{Hom}(a, a)$ and $\mathrm{Hom}(a[1], a)$ respectively.

with the gauge group generated by transformations of the form (2.27), which in our case become:

$$\begin{aligned}
\delta\phi^{(1)}_{aa} &= -d\alpha^{(0)}_{aa} - [\phi^{(1)}_{aa}, \alpha^{(0)}_{aa}]_\bullet - \phi^{(1+n)}_{ba} \bullet \alpha^{(-n)}_{ab} - \alpha^{(n)}_{ba} \bullet \phi^{(1-n)}_{ab} \\
\delta\phi^{(1)}_{bb} &= -d\alpha^{(0)}_{bb} - [\phi^{(1)}_{bb}, \alpha^{(0)}_{bb}]_\bullet - \phi^{(1-n)}_{ab} \bullet \alpha^{(n)}_{ba} - \alpha^{(-n)}_{ab} \bullet \phi^{(1+n)}_{ba} \\
\delta\phi^{(1+n)}_{ba} &= -d\alpha^{(n)}_{ba} - \phi^{(1+n)}_{ba} \bullet \alpha^{(0)}_{bb} - \alpha^{(0)}_{aa} \bullet \phi^{(1+n)}_{ba} - \phi^{(1)}_{aa} \bullet \alpha^{(n)}_{ba} - \alpha^{(n)}_{ba} \bullet \phi^{(1)}_{bb} \\
\delta\phi^{(1-n)}_{ab} &= -d\alpha^{(-n)}_{ab} - \phi^{(1-n)}_{ab} \bullet \alpha^{(0)}_{aa} - \alpha^{(0)}_{bb} \bullet \phi^{(1-n)}_{ab} - \phi^{(1)}_{bb} \bullet \alpha^{(-n)}_{ab} - \alpha^{(-n)}_{ab} \bullet \phi^{(1)}_{aa} \,,
\end{aligned} \tag{6.6}$$

with a parameter $\alpha = \begin{bmatrix} \alpha^{(0)}_{aa} & \alpha^{(n)}_{ba} \\ \alpha^{(-n)}_{ab} & \alpha^{(0)}_{bb} \end{bmatrix}$, where forms of rank outside of the range $0, \ldots, 3$ are of course vanishing.

As in (6.2), the graded commutator is now taken with respect to the boundary product $\bullet$ and worldsheet charge $|\cdot|$. For $n = 0$, this equation describes deformations of the block-diagonal flat connection $A = A_a \oplus A_a$ to a flat connection on the bundle $E \oplus E$ (which need not have block-diagonal form). However, the moduli space of solutions of (6.5) for $0 < |n| < 3$ contains slices of the *extended* moduli space of flat connections on E. For example a graded D-brane system with $n = 1$ contains deformations generated by $H^0(\mathrm{End}(E))$ and $H^2(\mathrm{End}(E))$, while for $n = 2$ we obtain deformations generated by $H^3(\mathrm{End}(E))$. Hence *one represents extended deformations of E through condensation of boundary condition changing operators in a pair of graded topological D-branes based on E* (figure 2).

A complete description of the extended moduli space requires consideration of all graded branes $a[n]$ with $n \in \mathbb{Z}$, and is discussed at that level of generality in [6]; the condensates resulting from a given graded D-brane pair only describe various slices through $\mathcal{M}_e$. One approach to the physical significance of points in $\mathcal{M}_e - \mathcal{M}$ is to study the quantum dynamics of the associated string field theories. A first step in this direction is the BV analysis carried out in this paper. This interpretation of graded D-brane condensates is the main reason for our interest in graded Chern-Simons theory.

6.3 Acyclic composites

The composite resulting from a condensation process can happen to be *acyclic*, i.e. the cohomology of the shifted BRST operator d' may vanish in all degrees. In this case, the theory expanded around the composite contains only BRST trivial states, and the resulting point in the (extended) moduli space can be interpreted as a closed string vacuum.

Observation. The axioms of string field theory imply that the bilinear form $\langle\cdot,\cdot\rangle$ induces a perfect pairing between the BRST cohomology groups $H^k_{d'}(\mathcal{H})$ and $H^{3-k}_{d'}(\mathcal{H})$ for any worldsheet U(1) charge k. In particular, acyclicity of a composite can be established by checking vanishing of only half of the cohomology groups $H^k_{d'}(\mathcal{H})$.

6.4 Acyclic composites for $n = 1$

6.4.1 Assumptions

In this subsection we discuss a particular class of condensation processes which is somewhat similar to tachyon condensation in bosonic string theory in that it produces acyclic composites. Condensation processes of the type discussed below can take place for a graded D-brane pair satisfying the conditions:

1. the relative grade of the pair is $n = 1$
2. E_a and E_b are isomorphic *as flat bundles.*
3. $H^1(\mathrm{End}(E_a)) = H^1(\mathrm{End}(E_b)) = H^1(\mathrm{Hom}(E_a, E_b)) = H^2(\mathrm{Hom}(E_a, E_b)) = 0.$

The argument presented below is closely related to ideas of [14], though our physical realization seems to be different.

It is not hard to see that condition (2) amounts to the existence of a *covariantly constant* isomorphism between E_a and E_b. This means a bundle isomorphism $f : E_a \to E_b$ which satisfies $df = 0$[21] (remember that the differential d on $\mathrm{Hom}(E_a, E_b)$ is coupled to the background flat connections A_a and A_b). The constraints on the first cohomology of E_a and E_b mean that these flat connections possess no deformations.

6.4.2 Construction of acyclic composites

With the assumptions discussed above, one can construct a solution ϕ of the string field equations of motion which leads to an acyclic D-brane composite. Indeed, let us look for a solution of the form $\phi = \begin{bmatrix} 0 & 0 \\ \phi^{(0)}_{ab} & 0 \end{bmatrix}$. In this case, the equations of motion $d\phi + \phi \bullet \phi = 0$

[21]It is not hard to see that such a map gives an isomorphism of the underlying flat structures. In the language of flat connections, the condition $df = 0$ implies that f induces a gauge transformation which takes A_a to A_b (one obtains $A_b = fA_af^{-1} - (df)f^{-1}$ upon choosing local frames for E_a and E_b). Alternately, f gives an isomorphism between flat trivializations of E_a and E_b.

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reduce to $d\phi_{ab}^{(0)} = 0$, which says that $\phi_{ab}^{(0)}$ is covariantly constant section of $\mathrm{Hom}(E_a, E_b)$. We shall further require that $\phi_{ab}^{(0)}$ is a bundle isomorphism, which is possible in view of our second assumption. Condensation of the boundary condition changing state $\phi_{ab}^{(0)}$ leads to a new D-brane background which can be viewed as composite of the D-branes a and b. The theory expanded around this background is endowed with the shifted BRST operator given by $d'_\phi u = d\phi + [\phi, u]_\bullet$. We wish to show that d'_ϕ has trivial cohomology. Since $n = 1$, the total boundary space of our D-brane system has non-vanishing components of charge $k = -1, 0, 1, 2, 3$ and 4. Moreover, Poincaré duality identifies the worldsheet BRST cohomologies in degrees k and $3 - k$, so that it suffices to show vanishing of $H^k_{d'}(\mathcal{H})$ for $k = -1, 0$ and 1.

6.4.3 Proof of acyclicity

Vanishing of $H^{-1}_{d'_\phi}(\mathcal{H})$. A state of charge -1 has the form $u = \begin{bmatrix} 0 & u_{ba}^{(0)} \\ 0 & 0 \end{bmatrix}$. In this case, the BRST closure condition $d'_\phi u = 0 \Leftrightarrow du + \phi \bullet u + u \bullet \phi = 0$ is equivalent with:

$$u_{ba}^{(0)} \bullet \phi_{ab}^{(0)} = 0\,, \qquad du_{ba}^{(0)} = 0\,, \qquad \phi_{ab}^{(0)} \bullet u_{ba}^{(0)} = 0\,. \tag{6.7}$$

Since $\phi_{ab}^{(0)}$ is a bundle isomorphism, this is equivalent with $u_{ba}^{(0)} = 0$ i.e. $u = 0$, which shows that $H^{-1}_{d'_\phi}(\mathcal{H}) = 0$.

Vanishing of $H^0_{d'_\phi}(\mathcal{H})$. Given a charge zero state $u = \begin{bmatrix} u_{aa}^{(0)} & u_{ba}^{(1)} \\ 0 & u_{bb}^{(0)} \end{bmatrix}$, the BRST closure condition $d'_\phi u = 0 \Leftrightarrow du + \phi \bullet u - u \bullet \phi = 0$ reads:

$$du_{aa}^{(0)} = u_{ba}^{(1)} \bullet \phi_{ab}^{(0)}\,, \qquad du_{ba}^{(1)} = 0\,, \qquad \phi_{ab}^{(0)} \bullet u_{aa}^{(0)} = u_{bb}^{(0)} \bullet \phi_{ab}^{(0)}\,, \qquad du_{bb}^{(0)} = -\phi_{ab}^{(0)} \bullet u_{ba}^{(1)}\,. \tag{6.8}$$

On the other hand, the exactness condition $u = d'_\phi \alpha = d\alpha + \phi \bullet \alpha + \alpha \bullet \phi$ (for an element $\alpha = \begin{bmatrix} 0 & \alpha_{ba}^{(0)} \\ 0 & 0 \end{bmatrix}$ of charge -1) is equivalent with:

$$u_{aa}^{(0)} = \alpha_{ba}^{(0)} \bullet \phi_{ab}^{(0)}\,, \qquad u_{ba}^{(1)} = d\alpha_{ba}^{(0)}\,, \qquad u_{bb}^{(0)} = \phi_{ab}^{(0)} \bullet \alpha_{ba}^{(0)}\,. \tag{6.9}$$

The assumption $H^2(\mathrm{Hom}(E_a, E_b)) = 0 \Leftrightarrow H^1(\mathrm{Hom}(E_b, E_a)) = 0$ allows us to solve the second equation of (6.8) in the form:

$$u_{ba}^{(1)} = d\alpha_{ba}^{(0)}\,, \tag{6.10}$$

with $\alpha_{ba}^{(0)}$ a section of $\mathrm{Hom}(E_b, E_a)$, determined up to

$$\alpha_{ba}^{(0)} \to \alpha_{ba}^{(0)} + f_{ba}^{(0)}\,, \tag{6.11}$$

where $f_{ba}^{(0)}$ is a *covariantly constant* section of $\mathrm{Hom}(E_b, E_a)$. Upon substituting (6.10) into the first and fourth equations of (6.8), these can be solved as:

$$u_{aa}^{(0)} = \alpha_{ba}^{(0)} \bullet \phi_{ab}^{(0)} + f_{aa}^{(0)} \tag{6.12}$$

and

$$u_{bb}^{(0)} = \phi_{ab}^{(0)} \bullet \alpha_{ba}^{(0)} + f_{bb}^{(0)} , \tag{6.13}$$

where $f_{aa}^{(0)}$ and $f_{bb}^{(0)}$ are covariantly constant sections of $\mathrm{End}(E_a)$ and $\mathrm{End}(E_b)$ and we used $d\phi_{ab}^{(0)} = 0$. Using these expressions in the third equation of (6.8) gives:

$$\phi_{ab}^{(0)} \bullet f_{aa}^{(0)} = f_{bb}^{(0)} \bullet \phi_{ab}^{(0)} , \tag{6.14}$$

a condition which can be satisfied upon choosing $f_{bb}^{(0)} = \phi_{ab}^{(0)} f_{aa}^{(0)} (\phi_{ab}^{(0)})^{-1}$ (this is covariantly constant since each of the factors is). With this choice, both $f_{aa}^{(0)}$ and $f_{bb}^{(0)}$ can be eliminated from (6.12) and (6.13) upon using the freedom (6.11) to redefine:

$$\alpha_{ba}^{(0)} \rightarrow \alpha_{ba}^{\prime(0)} := \alpha_{ba}^{(0)} + f_{aa}^{(0)} \bullet (\phi_{ab}^{(0)})^{-1} . \tag{6.15}$$

This shows that equations (6.10), (6.12) and (6.13) can be brought to the form (6.9). We conclude that $H^0_{d'_\phi}(\mathcal{H})$ vanishes as well.

Vanishing of $H^1_{d'_\phi}(\mathcal{H})$. If $u = \begin{bmatrix} u_{aa}^{(1)} & u_{ba}^{(2)} \\ u_{ab}^{(0)} & u_{bb}^{(1)} \end{bmatrix}$ is a state with $|u| = 1$, then the BRST closure condition $d'_\phi u = 0 \Leftrightarrow du + [\phi, u]_\bullet = 0$ is equivalent with the system:

$$\begin{aligned} du_{aa}^{(1)} + u_{ba}^{(2)} \bullet \phi_{ab}^{(0)} &= 0 \\ du_{ba}^{(2)} &= 0 \\ du_{ab}^{(0)} + \phi_{ab}^{(0)} \bullet u_{aa}^{(1)} + u_{bb}^{(1)} \bullet \phi_{ab}^{(0)} &= 0 \\ du_{bb}^{(1)} + \phi_{ab}^{(0)} \bullet u_{ba}^{(2)} &= 0 . \end{aligned} \tag{6.16}$$

Using the assumption $H^2(\mathrm{Hom}(E_b, E_a)) = 0$, the second equation can be solved as:

$$u_{ba}^{(2)} = d\alpha_{ba}^{(1)} , \tag{6.17}$$

for some one form $\alpha_{ba}^{(1)}$ with coefficients in $\mathrm{Hom}(E_b, E_a)$. Upon substituting this into the first equation and using $d\phi_{ab}^{(0)} = 0$, one obtains:

$$u_{aa}^{(1)} = d\alpha_{aa}^{(0)} - \alpha_{ba}^{(1)} \bullet \phi_{ab}^{(0)} , \tag{6.18}$$

for some section $\alpha_{aa}^{(0)}$ of $\mathrm{End}(E_a)$. To arrive at this equation, we made use of the assumption $H^1(\mathrm{End}(E_a)) = 0$. The section $\alpha_{aa}^{(0)}$ is determined up to transformations of the form:

$$\alpha_{aa}^{(0)} \rightarrow \alpha_{aa}^{(0)} + f_{aa}^{(0)} , \tag{6.19}$$

with $f_{aa}^{(0)}$ a *covariantly constant* (flat) section of $\mathrm{End}(E_a)$. Using the solution for $u_{ba}^{(2)}$ and the condition $d\phi_{ab}^{(0)} = 0$, we can similarly solve the last equation of (6.16):

$$u_{bb}^{(1)} = d\alpha_{bb}^{(0)} + \phi_{ab}^{(0)} \bullet \alpha_{ba}^{(1)} , \tag{6.20}$$

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where we used the assumption $H^1(\mathrm{End}(E_b)) = 0$. Finally, combining (6.17) and (6.18) allows us to solve the third equation in (6.16):

$$u_{ab}^{(0)} = \phi_{ab}^{(0)} \bullet \alpha_{aa}^{(0)} - \alpha_{bb}^{(0)} \bullet \phi_{ab}^{(0)} + f_{ab}^{(0)} , \tag{6.21}$$

with $f_{ab}^{(0)}$ a covariantly constant section of $\mathrm{Hom}(E_a, E_b)$.

Since both sections $f_{ab}^{(0)}$ and $\phi_{ab}^{(0)}$ of $\mathrm{Hom}(E_a, E_b)$ are covariantly constant, the section $g_{aa}^{(0)} := (\phi_{ab}^{(0)})^{-1} \circ f_{ab}^{(0)}$ of $\mathrm{End}(E_a)$ satisfies $dg_{aa}^{(0)} = 0$. Upon using the ambiguity (6.19), we can therefore absorb $f_{aa}^{(0)}$ in $\alpha_{aa}^{(0)}$ by defining:

$$\alpha_{aa}^{\prime(0)} = \alpha_{aa}^{(0)} + g_{aa}^{(0)} . \tag{6.22}$$

This allows us to re-write (6.21) in the form:

$$u_{ab}^{(0)} = \phi_{ab}^{(0)} \bullet \alpha_{aa}^{\prime(0)} - \alpha_{bb}^{(0)} \bullet \phi_{ab}^{(0)} . \tag{6.23}$$

Hence we can assume without loss of generality that $\alpha_{aa}^{(0)}$ has been chosen such that:

$$u_{ab}^{(0)} = \phi_{ab}^{(0)} \bullet \alpha_{aa}^{(0)} - \alpha_{bb}^{(0)} \bullet \phi_{ab}^{(0)} . \tag{6.24}$$

The last step is to define the state $\alpha = \left[\begin{pmatrix} \alpha_{aa}^{(0)} & \alpha_{ba}^{(1)} \\ 0 & \alpha_{bb}^{(0)} \end{pmatrix}\right]$, which has charge zero. It is then easy to check that equations (6.17), (6.18) and (6.20) and (6.24) are equivalent with the condition:

$$u = d'_\phi \alpha = d\alpha + [\phi, \alpha]_\bullet . \tag{6.25}$$

Thus every d'_ϕ-closed degree one state is d'_ϕ-exact, which means that the BRST complex of d'_ϕ is also trivial in degree one. This shows that the composite obtained by condensation of $\phi_{ab}^{(0)}$ is acyclic.

Note that the boundary condition changing state $\phi_{ab}^{(0)}$ is physical, since it is BRST-closed and has charge 1. In fact, the ('unshifted') BRST cohomology $H^1_d(\mathcal{H}_{ab})$ in degree one coincides with the space $H^0_d(\mathrm{Hom}(E_a, E_b))$ of covariantly constant sections of $\mathrm{Hom}(E_a, E_b)$.

Observations.

1. We wish to stress that one does *not* need to consider the extended action (3.34), nor the extended boundary space $\mathcal{H}_e$, in order to understand condensation processes.

2. We mention that the assumption $E_a \approx E_b$ is necessary only for establishing acyclicity in the sector with U(1) charge -1. In the sector with vanishing U(1) charge, this condition can be avoided by choosing $f_{aa}^{(0)} = g_{ba}^{(0)} \bullet \phi_{ab}^{(0)}$ and $f_{bb}^{(0)} = \phi_{ab}^{(0)} \bullet g_{ba}^{(0)}$ in equations (6.12) and (6.13), where g_{ba} is a covariantly constant section of $\mathrm{Hom}(E_b,\ E_a)$. Then equation (6.14) is automatically satisfied and one shifts $\alpha_{ba}^{(0)}$ to $\alpha_{ba}^{\prime(0)} = \alpha_{ba}^{(0)} + g_{ba}^{(0)}$ instead of equation (6.15). Hence vanishing of $H^0_{d'}(\mathcal{H})$ can be established without requiring invertibility of $\phi_{ab}^{(0)}$. A similar modification of the argument is possible when showing vanishing of $H^1_{d'}(\mathcal{H})$; namely one chooses $f_{ab}^{(0)}$ to be of the form $f_{ab}^{(0)} = \phi_{ab}^{(0)} \bullet g_{aa}^{(0)}$, with $g_{aa}^{(0)}$ a covariantly constant section of $\mathrm{End}(E_a)$. However, the condition that ϕ_{ab} is invertible is crucial for assuring vanishing of $H^{-1}_{d'}(\mathcal{H})$.

3. The fact that a D-brane pair satisfying our conditions can condense to an acyclic composite gives some justification for viewing these systems as 'topological brane-antibrane pairs'. However, we were able to establish the existence of such processes only under restrictive topological assumptions on the original background. To understand just how restrictive our hypotheses are, consider the case of two singly-wrapped graded D-branes, for which $E_a = E_b = \mathcal{O}_L$, the trivial line bundle over L (endowed with the trivial flat connection $A_a = A_b = 0$). In this situation, our argument requires vanishing of the cohomology group $H^1(L) \approx H^2(L)$, which in particular means that the cycle L must be a rational homology sphere. It is well-known that Calabi-Yau threefolds contain a wealth of special lagrangian cycles which do not satisfy this condition (for example, special lagrangian 3-tori, which according to the conjecture of [44] should give a fibration of X if X admits a geometric mirror). Our argument does not apply to such cycles, even in the singly-wrapped case. It is an interesting problem to study composite formation in this more general situation.

7. Direct construction of the master action for D-brane pairs

In this section we give a direct construction of the classical BV actions for the string field theories of graded D-brane pairs. This will allow us to recover the extended theory (3.34) by applying the more familiar cohomological formalism. Moreover, we shall give a BV level proof that there are precisely six inequivalent classical BV actions, namely those associated to the relative gradings $n = 0, \pm 1, \pm 2$ and $(n \geq 3,\ n \leq -3)$, thus leading to a $\mathbb{Z}_6$ multiplicity of such systems. This gives the string field theory realization of an observation made in [1].

As already mentioned at the end of section 4, the BV algorithm prolongs a basis for the space of classical fields to a basis $\mathcal{B} = (e^{*\alpha}, e_\alpha)$ for the space of extended fields, which is Darboux for the underlying odd symplectic form. Since the coefficients of the latter are particularly simple in this basis, the BV bracket takes the form (4.52).

In this section we apply the BV algorithm in a component approach, and carry out the computations in terms of the usual wedge product of forms (which we shall write as juxtaposition in order to simplify notation), rather than in terms of the extended boundary product $*$ used in the previous section. This makes for an explicit presentation, at the cost of introducing somewhat complicated formulae. A more synthetic description (which uses the results of the geometric formalism in order to simplify certain steps) is given in the next section. As discussed in section 5, it is enough to consider the cases with relative grading $n \equiv n_b - n_a \geq 0$, since the rest can be obtained by reversing the roles of a and b.[22]

[22] Two D-brane pairs with relative gradings n and $-n$ are related by the conjugation operation discussed in [6]. While this is a 'symmetry' at the classical level, two systems related in this manner should not be identified. The conjugation symmetry of [6] is akin to the charge conjugation of particle physics, and conjugated configurations must be considered physically distinct.

7.1 The BV action for $n = 1$

For the case $n = 1$, the classical fields are two one-forms, a two-form and a zero-form:

$$\phi = \begin{pmatrix} \phi^{(1)} & \phi^{(2)} \\ \phi^{(0)} & \phi'^{(1)} \end{pmatrix} . \tag{7.1}$$

In terms of the usual wedge product (3.23), the classical action (5.10) reads:

$$\begin{aligned} S(\phi) = & \int_L \mathrm{tr}_a \left[\frac{1}{2} \left(\phi^{(1)} d\phi^{(1)} - \phi^{(2)} d\phi^{(0)} \right) + \frac{1}{3} \left(\phi^{(1)}\phi^{(1)}\phi^{(1)} + \phi^{(1)}\phi^{(2)}\phi^{(0)} \right) \right] - \\ & - \int_L \mathrm{tr}_b \left[\frac{1}{2} \left(\phi'^{(1)} d\phi'^{(1)} - \phi^{(0)} d\phi^{(2)} \right) + \frac{1}{3} \left(\phi'^{(1)}\phi'^{(1)}\phi'^{(1)} + \phi'^{(1)}\phi^{(0)}\phi^{(2)} \right) \right] . \end{aligned} \tag{7.2}$$

Upon substituting $\alpha = -C_1\lambda$ into the gauge transformations (6.4), where λ is a Grassmann-odd constant parameter and:

$$C_1 = \begin{pmatrix} c_1^{(0)} & c_1^{(1)} \\ 0 & c'^{(0)}_1 \end{pmatrix} , \tag{7.3}$$

is the corresponding matrix of ghosts, we find the BRST transformations of the physical fields:

$$\begin{aligned} \delta^{(1)}\phi^{(1)} &= \left[dc_1^{(0)} + [\phi^{(1)}, c_1^{(0)}] - c_1^{(1)}\phi^{(0)} \right] \lambda \\ \delta^{(1)}\phi^{(2)} &= \left[dc_1^{(1)} + (\phi^{(1)}c_1^{(1)} + \phi^{(2)}c'^{(0)}_1 - c_1^{(0)}\phi^{(2)} + c_1^{(1)}\phi'^{(1)}) \right] \lambda \\ \delta^{(1)}\phi^{(0)} &= (\phi^{(0)}c_1^{(0)} - c'^{(0)}_1\phi^{(0)})\lambda \\ \delta^{(1)}\phi'^{(1)} &= \left[dc'^{(0)}_1 + [\phi'^{(1)}, c'^{(0)}_1] - \phi^{(0)}c_1^{(1)} \right] \lambda . \end{aligned} \tag{7.4}$$

Requiring nilpotence of $\delta^{(1)}$ leads to the following ghost transformations:

$$\delta^{(1)}c_1^{(0)} = c_1^{(0)}c_1^{(0)}\lambda , \qquad \delta^{(1)}c'^{(0)}_1 = c'^{(0)}_1c'^{(0)}_1\lambda , \qquad \delta^{(1)}c_1^{(1)} = (c_1^{(0)}c_1^{(1)} + c_1^{(1)}c'^{(0)}_1)\lambda . \tag{7.5}$$

It is easy to see that the gauge transformations (6.6) vanish on-shell for a particular choice of parameters. Oversimplifying, consider the transformation[23] $\delta\phi^{(2)} = -d\alpha^{(1)} - \phi^{(1)}\alpha^{(1)} + \alpha^{(1)}\phi'^{(1)}$ and take $\alpha^{(1)}$ to be of the form $\alpha^{(1)} = -(d\beta^{(0)} + \phi^{(1)}\beta^{(0)} + \beta^{(0)}\phi'^{(1)})$, with $\beta^{(0)}$ a section of $\mathrm{Hom}(E_b, E_a)$. Substituting this into the gauge transformation, one finds that $\delta\phi^{(2)}$ vanishes when $\phi^{(1)}$ and $\phi'^{(1)}$ satisfy their equations of motion (in this exemplification we ignored the role of $\alpha^{(0)}$ and $\alpha'^{(0)}$ in the BRST transformation). It

[23] We remind the reader that the cohomological (BRST) approach extends the gauge algebra $\mathbf{g}$ of section 2.2.2 to the jet bundle associated with our fields. Thus, one views the various partial derivatives of a generator as being independent quantities. This is why we can consider 'differential zero modes', i.e. expressions involving the exterior derivative of some generator. The jet bundle extension is a standard device for describing the on-shell algebraic structure of gauge transformations and observables, and forms the heart of the modern approach to BRST quantization. We refer the reader to [33] for an elegant review of the jet bundle formalism and further references.

follows that our gauge algebra is *reducible*. In such a situation, one must introduce ghosts for ghosts. Accordingly, we consider a second generation ghost:

$$C_2 = \begin{pmatrix} 0 & c_2^{(0)} \\ 0 & 0 \end{pmatrix}, \tag{7.6}$$

which allows us to extend the BRST transformation of $c_1^{(1)}$ by adding:

$$\delta_2 c_1^{(1)} = \left(dc_2^{(0)} + \phi^{(1)} c_2^{(0)} - c_2^{(0)} \phi'^{(1)} \right) \lambda\,. \tag{7.7}$$

Allowing δ_2 to act trivially on the physical fields ϕ and requiring nilpotence of $\delta^{(2)} \equiv \delta^{(1)} + \delta_2$ leads to the following corrections to the transformations of $c_1^{(0)}$ and $c'^{(0)}_1$:

$$\delta_2 c_1^{(0)} = c_2^{(0)} \phi^{(0)} \lambda\,, \qquad \delta_2 c'^{(0)}_1 = \phi^{(0)} c_2^{(0)} \lambda \tag{7.8}$$

and specifies the BRST transformation of the second generation ghost:

$$\delta_2 c_2^{(0)} = (-c_1^{(0)} c_2^{(0)} + c_2^{(0)} c'^{(0)}_1) \lambda\,. \tag{7.9}$$

Since there are no further zero modes, the gauge algebra is first order reducible and the full BRST transformations are given by:

$$\delta \equiv \delta^{(2)} = \delta^{(1)} + \delta_2 \tag{7.10}$$

where we let $\delta^{(1)}$ act trivially on second generation ghosts: $\delta^{(1)} c_2^{(0)} = 0$.

The BV construction now introduces antifields Φ^*, C_1^* and C_2^* for each of the fields ϕ, C_1 and C_2:

$$\Phi^* = \begin{pmatrix} \phi^{*(2)} & \phi^{*(3)} \\ \phi^{*(1)} & \phi'^{*(2)} \end{pmatrix}, \qquad {C_1}^* = \begin{pmatrix} c_1^{*(3)} & 0 \\ c_1^{*(2)} & c'^{*(3)}_1 \end{pmatrix}, \qquad {C_2}^* = \begin{pmatrix} 0 & 0 \\ c_2^{*(3)} & 0 \end{pmatrix}. \tag{7.11}$$

Note that the antifield of each matrix block sits in the transposed position in the full matrix, for example $(\phi^{(1)}, \phi^{*(2)})$, $(\phi'^{(1)}, \phi'^{*(2)})$, $(\phi^{(2)}, \phi^{*(1)})$ and $(\phi^{(0)}, \phi^{*(3)})$ form field-antifield pairs. The classical action S is extended by adding a term of the form $S_1 = \mathrm{tr}(\Phi^* \delta\phi/\lambda + C_1^* \delta C_1/\lambda + \ldots)$. The Grassmann parities of antifields are chosen such that S_1 is even: $g(\text{field}) + g(\text{antifield}) = \hat{1}$ (mod 2).

Following this procedure we write the first order action:

$$\begin{aligned} S_1 = \int_L \mathrm{tr}_a \Big[& \phi^{*(2)} \left(dc_1^{(0)} + [\phi^{(1)}, c_1^{(0)}] - c_1^{(1)} \phi^{(0)} \right) + \\ & + c_1^{*(3)} \left(c_1^{(0)} c_1^{(0)} + c_2^{(0)} \phi^{(0)} \right) + \phi^{*(3)} \left(\phi^{(0)} c_1^{(0)} - c'^{(0)}_1 \phi^{(0)} \right) \Big] + \\ + \int_L \mathrm{tr}_b \Big[& \phi^{*(1)} \left(dc_1^{(1)} + (\phi^{(1)} c_1^{(1)} + \phi^{(2)} c'^{(0)}_1 - c_1^{(0)} \phi^{(2)} + c_1^{(1)} \phi'^{(1)}) \right) + \\ & + \phi'^{*(2)} \left(dc'^{(0)}_1 + [\phi'^{(1)}, c'^{(0)}_1] - \phi^{(0)} c_1^{(1)} \right) + \\ & + c_1^{*(2)} \left(c_1^{(0)} c_1^{(1)} + c_1^{(1)} c'^{(0)}_1 + dc_2^{(0)} + \phi^{(1)} c_2^{(0)} - c_2^{(0)} \phi'^{(1)} \right) + \\ & + c'^{*(3)}_1 \left(c'^{(0)}_1 c'^{(0)}_1 + \phi^{(0)} c_2^{(0)} \right) + c_2^{*(3)} \left(-c_1^{(0)} c_2^{(0)} + c_2^{(0)} c'^{(0)}_1 \right) \Big]. \end{aligned} \tag{7.12}$$

Higher order terms in the antifield expansion $S_{BV} = S_0 + S_1 + S_2 + \cdots$ are constructed from the requirement that S_{BV} satisfies the master equation. Since vanishing of $\{S, S_1\}$ is guaranteed by gauge invariance of the classical action, the next step is to compute the BV bracket $\{S_1, S_1\}$ and find a non-vanishing result, linear in antifields.[24] Thus, we have to supplement S_1 by a further term S_2 quadratic in antifields, such that $\{S_1, S_1\} + 2\{S, S_2\} = 0$. It is not very hard to check that the choice:

$$S_2 = \int_L \mathrm{tr}_a \left(c_2^{(0)} \phi^{*(1)} \phi^{*(2)} - c_2^{(0)} \phi'^{*(2)} \phi^{*(1)} \right), \tag{7.13}$$

assures that the BV action:

$$S_{BV} = S + S_1 + S_2 \tag{7.14}$$

obeys the master equation. Some details of the relevant computation are given in appendix C.

To show that the BV action coincides with the coordinate-free expression (3.34), we set:

$$\begin{aligned}\hat{\phi} &= \begin{pmatrix} c_1^{(0)} + \phi^{(1)} - \phi^{*(2)} + c_1^{*(3)} & c_2^{(0)} + c_1^{(1)} + \phi^{(2)} - \phi^{*(3)} \\ \phi^{(0)} + \phi^{*(1)} + c_1^{*(2)} + c_2^{*(3)} & c'^{(0)}_1 + \phi'^{(1)} + \phi'^{*(2)} - c'^{*(3)}_1 \end{pmatrix} \\ &= c_2 + c_1 + \phi + \phi^* + c_1^* + c_2^*\,, \end{aligned} \tag{7.15}$$

where we defined:

$$\phi^* = \begin{pmatrix} -\phi^{*(2)} & -\phi^{*(3)} \\ \phi^{*(1)} & \phi'^{*(2)} \end{pmatrix}, \qquad {c_1}^* = \begin{pmatrix} c_1^{*(3)} & 0 \\ c_1^{*(2)} & -c'^{*(3)}_1 \end{pmatrix}, \tag{7.16}$$

and:

$$c_1 = C_1\,, \qquad c_2 = C_2\,, \qquad {c_2}^* = C_2^*\,, \tag{7.17}$$

notations which will be useful in the next section. Counting worldsheet U(1) charges and Grassmann parities shows that $\hat{\phi}$ is an element of $M = \mathcal{H}^1_e$. Relation (7.15) can be viewed as the expression of the vector $\hat{\phi}$ in the particular linear coordinates on M built by the BV algorithm.

One can check by direct computation that substitution of (7.15) into the extended action (3.34) recovers the expanded form (7.14) upon expressing everything in terms of the usual wedge product. This shows that the BV action (7.14) is simply the form of the extended action in the particular linear coordinates (7.15). To show equivalence with the geometric formalism of section 4, we must also check that (7.15) gives Darboux coordinates for the odd symplectic form (4.66). As mentioned after eq. (4.67), the odd symplectic form is completely determined by its restriction to *even* vector fields, which can be identified with the form ω_0 of equation (3.41). It thus suffices to check that (3.41) reduces to Darboux

[24]In fact, the BRST transformation of $c_2^{(0)}$ can also be inferred from the requirement $\{S_1, S_1\}$ contains no terms which are independent of antifields.

form in the coordinates (7.15). This follows by direct computation upon substituting (7.15) into (3.41):[25]

$$\begin{aligned}\omega_0 = \mathrm{tr}_a\Big[&(\delta c_1^{*(3)}\wedge\delta c_1^{(0)} - \delta c_1^{(0)}\wedge\delta c_1^{*(3)}) + (\delta\phi^{*(2)}\wedge\delta\phi^{(1)} - \delta\phi^{(1)}\wedge\delta\phi^{*(2)}) +\\ &+ (\delta c_2^{*(3)}\wedge\delta c_2^{(0)} - \delta c_2^{(0)}\wedge\delta c_2^{*(3)}) + (\delta c_1^{*(2)}\wedge\delta c_1^{(1)} - \delta c_1^{(1)}\wedge\delta c_1^{*(2)}) +\\ &+ (\delta\phi^{*(1)}\wedge\delta\phi^{(2)} - \delta\phi^{(2)}\wedge\delta\phi^{*(1)})\Big] +\\ +\mathrm{tr}_b\Big[&(\delta c'^{*(3)}_1\wedge\delta c'^{(0)}_1 - \delta c'^{(0)}_1\wedge\delta c'^{*(3)}_1) + (\delta\phi'^{*(2)}\wedge\delta\phi'^{(1)} - \delta\phi'^{(1)}\wedge\phi'^{*(2)}) +\\ &+ (\delta\phi^{*(3)}\wedge\delta\phi^{(0)} - \delta\phi^{(0)}\wedge\delta\phi^{*(3)})\Big]\,. \end{aligned} \quad (7.18)$$

We conclude that the homological construction for a D-brane pair with relative grading $n = 1$ agrees with the BV system discussed in section 4.

7.2 The BV action for $n = 2$

We next consider D-brane pairs with relative grading $n = 2$. For such systems, the classical fields are two one-forms and a three form:

$$\phi = \begin{pmatrix} \phi^{(1)} & \phi^{(3)} \\ 0 & \phi'^{(1)} \end{pmatrix}\,. \quad (7.19)$$

Since n is even, the relative grading plays no role in the boundary product $\bullet$, which reduces to the usual wedge product (3.23). The classical action is:

$$S(\phi) = \int_L \mathrm{tr}_a\left[\frac{1}{2}\phi^{(1)}d\phi^{(1)} + \frac{1}{3}\phi^{(1)}\phi^{(1)}\phi^{(1)}\right] + \int_L \mathrm{tr}_b\left[\frac{1}{2}\phi'^{(1)}d\phi'^{(1)} + \frac{1}{3}\phi'^{(1)}\phi'^{(1)}\phi'^{(1)}\right]. \quad (7.20)$$

Note that the field $\phi^{(3)}$ does not appear in the action, and thus its components define classical flat directions in the moduli space of the theory. We stress that, even though the action (7.20) is invariant with respect to the shift symmetry $\phi^{(3)} \to \phi^{(3)} + b^{(3)}$ (with $b^{(3)}$ an arbitrary 3-form valued in the bundle $\mathrm{Hom}(E_b, E_a)$), we do *not* include such symmetries in our gauge algebra. The reason is that a nonzero background value of $\phi^{(3)}$ corresponds to condensation of boundary condition changing states between the graded branes a and b, and thus leads to a *physically relevant* shift of the string vacuum. This forbids us to identify $\phi^{(3)}$ and $\phi^{(3)} + b^{(3)}$. This means that we cannot treat shifts of $\phi^{(3)}$ as *gauge* symmetries.

Since one of the classical fields is a massless 3-form, we expect first, second and third generation ghosts:

$$C_1 = \begin{pmatrix} c_1^{(0)} & c_1^{(2)} \\ 0 & c'^{(0)}_1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & c_2^{(1)} \\ 0 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & c_3^{(0)} \\ 0 & 0 \end{pmatrix}. \quad (7.21)$$

The BRST transformations of the classical fields read:

$$\begin{aligned} \delta^{(1)}\phi^{(1)} &= \left(dc_1^{(0)} + [\phi^{(1)}, c_1^{(0)}]\right)\lambda\,, \qquad \delta^{(1)}\phi'^{(1)} = \left(dc'^{(0)}_1 + [\phi'^{(1)}, c'^{(0)}_1]\right)\lambda \\ \delta^{(1)}\phi^{(3)} &= \left(dc_1^{(2)} - c_1^{(2)}\phi'^{(1)} + \phi^{(1)}c_1^{(2)} - c_1^{(0)}\phi^{(3)} + \phi^{(3)}c'^{(0)}_1\right)\lambda\,. \end{aligned} \quad (7.22)$$

[25]To obtain Darboux coordinates per se, one must choose bases for the spaces of bundle-valued forms corresponding to each block in the decomposition of our matrices of morphisms. We leave the details to the reader.

The transformations of first generation ghosts are derived by requiring nilpotence of $\delta^{(1)}$ on ϕ:

$$\delta^{(1)} c_1^{(2)} = \left(c_1^{(0)} c_1^{(2)} + c_1^{(2)} c'^{(0)}_1\right)\lambda\,, \qquad \delta^{(1)} c_1^{(0)} = c_1^{(0)} c_1^{(0)}\lambda\,, \qquad \delta^{(1)} c_1'^{(0)} = c_1'^{(0)} c'^{(0)}_1\lambda\,. \tag{7.23}$$

Once again, (7.22) has on-shell zero modes, which requires us to extend the BRST transformation of $c_1^{(2)}$ by adding the variation:

$$\delta_2 c_1^{(2)} = (dc_2^{(1)} + \phi^{(1)} c_2^{(1)} + c_2^{(1)}\phi'^{(1)})\lambda\,. \tag{7.24}$$

Allowing δ_2 to act trivially on the physical fields ($\delta_2\phi = 0$) and requiring nilpotence of $\delta^{(2)} = \delta^{(1)} + \delta_2$ implies that the second generation ghosts must transform as

$$\delta_2 c_2^{(1)} = (-c_1^{(0)} c_2^{(1)} + c_2^{(1)} c'^{(0)}_1)\lambda\,. \tag{7.25}$$

Equation (7.24) has further zero modes which lead to the third generation ghosts and to the transformations:

$$\delta_3 c_2^{(1)} = \left(dc_3^{(0)} - c_3^{(0)}\phi'^{(1)} + \phi^{(1)} c_3^{(0)}\right)\lambda\,, \qquad \delta_3 c_3^{(0)} = \left(c_1^{(0)} c_3^{(0)} + c_3^{(0)} c'^{(0)}_1\right)\lambda\,. \tag{7.26}$$

Since there are no further zero modes, the theory is second order reducible and we conclude that the full BRST transformations are given by:

$$\delta \equiv \delta^{(3)} = \delta^{(2)} + \delta_3 = \delta^{(1)} + \delta_2 + \delta_3 \tag{7.27}$$

where $\delta^{(1)}$ acts trivially on the second and third generation ghosts, δ_2 acts trivially on the physical fields and the third generation ghosts and δ_3 acts trivially on the physical fields and the first generation ghosts.

We next add the antifields:

$$\Phi^* = \begin{pmatrix} \phi^{*(2)} & 0 \\ \phi^{*(0)} & \phi'^{*(2)} \end{pmatrix},\ C_1^* = \begin{pmatrix} c_1^{*(3)} & 0 \\ c_1^{*(1)} & c'^{*(3)}_1 \end{pmatrix},\ C_2^* = \begin{pmatrix} 0 & 0 \\ c_2^{*(2)} & 0 \end{pmatrix},\ C_3^* = \begin{pmatrix} 0 & 0 \\ c_3^{*(3)} & 0 \end{pmatrix}, \tag{7.28}$$

and build the first order action S_1 as in the previous section.

Since $\{S_1, S_1\}$ does not vanish, the last step is to add a term S_2 quadratic in antifields, such that the classical master equation is satisfied. This leads to the full BV action:

$$\begin{aligned} S_{BV} = \int_L \mathrm{tr}_a \Big[& \frac{1}{2}\phi^{(1)} d\phi^{(1)} + \frac{1}{3}\phi^{(1)}\phi^{(1)}\phi^{(1)} + \phi^{*(2)}\left(dc_1^{(0)} + [\phi^{(1)},\, c_1^{(0)}]\right) + \\ & + c_1^{*(3)}\, c_1^{(0)} c_1^{(0)} - c_3^{(0)}\left(\phi'^{*(2)} c_1^{*(1)} - c_1^{*(1)}\phi^{*(2)} + \phi^{*(0)} c_1^{*(3)} - c'^{*(3)}_1\phi^{*(0)}\right) + \\ & + c_2^{(1)}\left(\phi'^{*(2)}\phi^{*(0)} + \phi^{*(0)}\phi^{*(2)}\right)\Big] + \\ + \int_L \mathrm{tr}_b \Big[& \frac{1}{2}\phi'^{(1)} d\phi'^{(1)} + \frac{1}{3}\phi'^{(1)}\phi'^{(1)}\phi'^{(1)} + \phi'^{*(2)}\left(dc'^{(0)}_1 + [\phi'^{(1)}, c'^{(0)}_1]\right) + \\ & + \phi^{*(0)}\left(dc_1^{(2)} - c_1^{(2)}\phi'^{(1)} + \phi^{(1)} c_1^{(2)} - c_1^{(0)}\phi^{(3)} + \phi^{(3)} c_1'^{(0)}\right) + \end{aligned}$$

$$
\begin{aligned}
&+ c_1^{*(1)} \left(c_1^{(0)} c_1^{(2)} + c_1^{(2)} c'^{(0)}_1 + d c_2^{(1)} + \phi^{(1)} c_2^{(1)} + c_2^{(1)} \phi'^{(1)} \right) + \\
&+ c_2^{*(2)} \left(d c_3^{(0)} - c_3^{(0)} \phi'^{(1)} + \phi^{(1)} c_3^{(0)} - c_1^{(0)} c_2^{(1)} + c_2^{(1)} c'^{(0)}_1 \right) + \\
&+ c'^{*(3)}_1 c'^{(0)}_1 c'^{(0)}_1 - c_3^{*(3)} \left(c_1^{(0)} c_3^{(0)} + c_3^{(0)} c'^{(0)}_1 \right) \Big] \,. \qquad (7.29)
\end{aligned}
$$

A brief discussion on the master equation can be found in appendix C.

This action can be cast into the form (3.34) provided that we identify the extended string field with:

$$
\begin{aligned}
\hat{\phi} &= \begin{pmatrix} c_1^{(0)} + \phi^{(1)} - \phi^{*(2)} + c_1^{*(3)} & c_3^{(0)} + c_2^{(1)} + c_1^{(2)} + \phi^{(3)} \\ -\phi^{*(0)} + c_1^{*(1)} - c_2^{*(2)} + c_3^{*(3)} & c'^{(0)}_1 + \phi'^{(1)} - \phi'^{*(2)} + c'^{*(3)}_1 \end{pmatrix} \\
&= c_3 + c_2 + c_1 + \phi + \phi^* + c_1^* + c_2^* + c_3^* \,, \qquad (7.30)
\end{aligned}
$$

where we defined:

$$
\phi^* = \begin{pmatrix} -\phi^{*(2)} & 0 \\ -\phi^{*(0)} & -\phi'^{*(2)} \end{pmatrix}, \qquad c_2^* = \begin{pmatrix} 0 & 0 \\ -c_2^{*(2)} & 0 \end{pmatrix} \qquad (7.31)
$$

and

$$
c_1 = C_1 \,, \qquad c_2 = C_2 \,, \qquad c_3 = C_3 \,, \qquad c_1^* = C_1^* \,, \qquad c_3^* = C_3^* \,. \qquad (7.32)
$$

We have $\deg \hat{\phi} = \hat{1}$, as required. Note that for $n = 2$, the graded trace str_e in (3.34) coincides with the ordinary trace. One checks again by direct computation that (7.30) gives Darboux coordinates for the odd symplectic form and that the extended action (3.34) reduces to (7.29) in this basis.

7.3 The BV action for $n = 0$

We next consider D-brane pairs with relative grading $n = 0$. In this case, the gauge algebra is irreducible and one needs only first generation ghosts. The physical fields and their ghosts are given by:

$$
\phi = \begin{pmatrix} \phi^{(1)} & \check{\phi}^{(1)} \\ \tilde{\phi}^{(1)} & \phi'^{(1)} \end{pmatrix}, \qquad c_1 = \begin{pmatrix} c_1^{(0)} & \check{c}_1^{(0)} \\ \tilde{c}_1^{(0)} & c'^{(0)}_1 \end{pmatrix}. \qquad (7.33)
$$

Because the relative shift vanishes, the product $\bullet$ is the usual wedge product of forms and the classical action is the CS action for $E_a \oplus E_b$. Axelrod and Singer [45] quantized the same classical theory, in the Faddeev-Popov approach. They expressed the gauged fixed action as an 'extended CS action', where the top form in the extended string field $\hat{\phi}$ is constrained to vanish. Here we show that the BV construction yields the same extended action, but for us the top form is unconstrained.[26]

The BV action for this system is:

$$
S_{BV} = \int_L \mathrm{tr}_{E_a \oplus E_b} \left[\frac{1}{2} \phi d\phi + \frac{1}{3} \phi\phi\phi + \phi^* \left(dc_1 + \phi c_1 - c_1 \phi \right) + c_1^* c_1 c_1 \right] , \qquad (7.34)
$$

[26]This BV action is of course well-known (see, for example, [46]).

where

$$\Phi^* = \begin{pmatrix} \phi^{*(2)} & \phi^{*(2)} \\ \phi^{*(2)} & \phi'^{*(2)} \end{pmatrix} \qquad \text{and} \qquad c_1^* = \begin{pmatrix} c_1^{*(3)} & c_1^{*(3)} \\ c_1^{*(3)} & c_1'^{*(3)} \end{pmatrix} \tag{7.35}$$

are the antifields associated with (7.33).

To present S_{BV} in the form (3.34), one defines the extended string field:

$$\hat{\phi} = c_1 + \phi + \phi^* + c_1^* \,, \tag{7.36}$$

where $\phi^* := -\Phi^*$. Upon substituting this into (3.34), we recover the BV action (7.34). It is also easy to check that (7.36) gives Darboux coordinates for the odd symplectic form.

7.4 The BV action for $n \geq 3$

For relative grading $n \geq 3$, the physical field contains two one-forms:

$$\phi = \begin{pmatrix} \phi^{(1)} & 0 \\ 0 & \phi'^{(1)} \end{pmatrix} \tag{7.37}$$

and the action appears as the direct sum or difference (according to the supertrace prescription) of two Chern-Simons terms:

$$S = \int_L \mathrm{tr}_a \left[\frac{1}{2}\phi^{(1)} d\phi^{(1)} + \frac{1}{3}(\phi^{(1)})^3\right] + (-1)^n \int_L \mathrm{tr}_b \left[\frac{1}{2}\phi'^{(1)} d\phi'^{(1)} + \frac{1}{3}(\phi'^{(1)})^3\right]. \tag{7.38}$$

Based on this form, it would naively seem that we can obtain the BV action simply as a sum or difference of the classical master actions associated with the two Chern-Simons theories, $S_{BV\,\mathrm{naive}} = S^a_{BV} + S^b_{BV}$, where, according to the discussion of the previous subsection, the BV actions for the two sectors have the extended Chern-Simons form. However, this conclusion does not take into account the physically correct form of the gauge algebra. Indeed, a direct sum construction of the BV action is only justified if *both* the action and the gauge algebra are of direct sum form.

A systematic approach to the analysis of pairs with $n \geq 3$ is afforded by the device of 'gauging zero' [48], which in our case amounts to formally applying the BRST procedure to forms of ranks higher than 3. For this, we shall pretend that an $(n+1)$-form is not identically vanishing in three dimensions,[27] and declare that the physical fields are

$$\phi = \begin{pmatrix} \phi^{(1)} & \phi^{(n+1)} \\ 0 & \phi'^{(1)} \end{pmatrix} . \tag{7.39}$$

[27] For the rigorous reader, we note that this procedure can be justified in the jet bundle approach [33, 47]. Consider the bundle $\mathcal{W} = \mathrm{End}(\mathbf{E}) = \mathrm{End}(E_a \oplus E_b)$, whose typical fiber we denote by W. One defines local coordinates x^i on the base manifold and u^a on the fiber. The infinite jet bundle $J^\infty\mathcal{W}$ is a prolongation of $\mathcal{W}$ to an infinite-dimensional vector bundle with coordinates $u^a_I = (u^a, u^a_{i_1}, u^a_{i_1 i_2}, \ldots)$ where $i_k \in \{1,2,3\}$ and the indices are symmetrized; the coordinates $u_I = u_{i_1 \ldots i_k}$, for a symmetric multi-index $I = (i_1 \ldots i_k)$, are viewed as formal partial derivatives of u_a and regarded as functionally independent. A section σ of the jet bundle decomposes as $\sigma = \sum_{k=0}^\infty \sigma_k$, according to the number k of formal derivatives; the component σ_0 defines a section of the unextended bundle $\mathcal{W}$. A section s of $\mathcal{W}$ (with components $u^a \circ s = s^a$) induces a section of $J^\infty\mathcal{W}$, the *associated jet* $j^\infty s$, which has the property $u^a_I \circ j^\infty s = \partial_{i_1}\partial_{i_2}\ldots\partial_{i_r} s^a$. This decomposes in a sum $j^\infty s = (j^\infty s)_0 + (j^\infty s)_1 + (j^\infty s)_2 + \cdots$. The zeroth component of the jet is the original section, $(j^\infty s)_0 = s$. One can also define forms on the infinite jet bundle: an element $\omega^{(r,s)}$ of the *variational bicomplex* $\Omega^{*,*}(J^\infty\mathcal{W})$ can be locally expressed in the basis $\theta^{a_1}_{I_1} \wedge \ldots \theta^{a_r}_{I_r} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$ where the *contact forms* $\theta^a_I = du^a_I - \sum_{i=1}^3 u^a_{Ii} dx^i$ span the vertical directions, and have the property $(j^\infty s)^*(\theta^a_I) = d(\partial_I \varphi^a) -$

We next perform the BV construction taking this as a starting point. Forms of ranks higher than 3 will be set to zero only in the final result.

Correspondingly, we have the BRST variations:

$$\begin{aligned} \delta^{(1)}\phi^{(1)} &= \left(dc_1^{(0)} + [\phi^{(1)},\, c_1^{(0)}]\right)\lambda\,, \qquad \delta^{(1)}\phi'^{(1)} = \left(dc'^{(0)}_1 + [\phi'^{(1)},\, c'^{(0)}_1]\right)\lambda \\ \delta^{(1)}\phi^{(n+1)} &= \left(dc_1^{(n)} + (-1)^{1+n}c_1^{(n)}\phi'^{(1)} + \phi^{(1)}c_1^{(n)} + \phi^{(n+1)}c'^{(0)}_1 - c_1^{(0)}\phi^{(n+1)}\right)\lambda\,. \end{aligned} \tag{7.40}$$

These extend to nilpotent transformations provided that we use the following variations of the ghosts:

$$\delta^{(1)}c_1^{(0)} = c_1^{(0)}c_1^{(0)}\lambda\,; \qquad \delta^{(1)}c'^{(0)}_1 = c'^{(0)}_1 c'^{(0)}_1\lambda\,; \qquad \delta^{(1)}c'^{(n)}_1 = (c_1^{(0)}c_1^{(n)} + c_1^{(n)}c'^{(0)}_1)\lambda\,. \tag{7.41}$$

One can write the following more synthetic form of the first level BRST transformations:

$$\delta^{(1)}\phi = (dC_1 + [\phi,\, C_1]_\bullet)\,\lambda\,, \qquad \text{with } C_1 = \begin{pmatrix} c_1^{(0)} & c_1^{(n)} \\ 0 & c'^{(0)}_1 \end{pmatrix}. \tag{7.42}$$

On the classical shell the transformation $\delta^{(1)}\phi^{(n+1)}$ has a residual local invariance given by

$$\delta_2 c_1^{(n)} = \left(dc_2^{(n-1)} + \phi^{(1)}c_2^{(n-1)} + (-1)^n c_2^{(n-1)}\phi'^{(1)}\right)\lambda\,, \tag{7.43}$$

where $c_2^{(n-1)}$ is the second generation ghost. With the notations introduced in sections 7.1 and 7.2, the full BRST transformations $\delta^{(2)} \equiv \delta^{(1)} + \delta_2$ at the second level coincide with $\delta^{(1)}$ when acting on $\phi^{(1)}$, $\phi'^{(1)}$, $c_1^{(0)}$, $c'^{(0)}_1$, and act on $c_1^{(n)}$ as:

$$\delta^{(2)}c_1^{(n)} = \left(dc_2^{(n-1)} + \phi^{(1)}c_2^{(n-1)} + (-1)^n c_2^{(n-1)}\phi'^{(1)} + c_1^{(0)}c_1^{(n)} + c_1^{(n)}c'^{(0)}_1\right)\lambda\,. \tag{7.44}$$

The transformations of second generation ghosts are determined by requiring nilpotence of (7.44). On the other hand, on-shell zero modes of (7.44) lead to ghosts of the next generation.

Since we want to discuss all cases with $n \geq 3$, we give an inductive proof that the BRST transformations of the k^{th} generation ghosts have the form:

$$\begin{aligned} \delta^{(k)}c_k^{(n-k+1)} &= \left(dc_{k+1}^{(n-k)} + \phi^{(1)}c_{k+1}^{(n-k)} + (-1)^{n-k+2}c_{k+1}^{(n-k)}\phi'^{(1)} + \right. \\ &\quad \left. + (-1)^{k+1}\, c_1^{(0)}c_k^{(n-k+1)} + c_k^{(n-k+1)}c'^{(0)}_1\right)\lambda\,. \end{aligned} \tag{7.45}$$

$(\partial_{I^i} s^a)dx^i = 0$. The space $\mathcal{H}$ of classical fields is the space of sections of $\mathcal{W}\otimes\Lambda^*T^*L$, which prolongs to the space $\Omega^{0,*}(J^\infty\mathcal{W})$ of *horizontal forms*. Hence given a classical field $\phi = \sum_{s=1}^3 \phi_{i_1\ldots i_s}dx^{i_1}\wedge\ldots\wedge dx^{i_s}$, we can define its jet $j^\infty(\phi) = \sum_{s=1}^3 j^\infty(\phi_{i_1\ldots i_s})dx^{i_1}\wedge\ldots\wedge dx^{i_s} \in \Omega^{0,*}(J^\infty\mathcal{W})$, whose zeroth component $j(\phi)_0$ coincides with ϕ. The trick of introducing forms of rank higher than three can be understood as working with elements of $\Omega^{*,*}(J^\infty\mathcal{W})$. For our purpose, a 'formal k-form' ω is an element of the variational bicomplex having total degree $p+q=k$. This can be decomposed as $\omega = \omega^{(k,0)}\oplus\omega^{(k-1,1)}\oplus\omega^{(k-2,2)}\oplus\omega^{(k-3,3)}$ and has a horizontal component $\omega^{(0,k)}$ if and only if the formal rank k lies between 0 and 3. A similar construction can be given for the various ghost generations, upon considering Grassmann-valued forms. At the end, we shall 'evaluate' all such forms, i.e. take the zeroth jet component of their horizontal projection, which is a (Grassmann-valued) section of the physical bundle $\mathcal{W}\otimes\Lambda^*T^*L$. This defines a BV field, i.e. an element of the extended boundary space $\mathcal{H}_e$. It is clear that 'evaluation' in this sense applied on ω will produce zero unless the formal rank k belongs to $0,\ldots,3$, in which case the result of evaluation will be the zeroth jet component of $\omega^{(0,k)}$, i.e. a 'true' k-form field. This recovers the procedure of [48].

Assume that (7.45) holds for the $k-1$ ghosts:

$$\begin{aligned}\delta^{(k-1)} c_{k-1}^{(n-k+2)} = \Big(& dc_k^{(n-k+1)} + \phi^{(1)} c_k^{(n-k+1)} + (-1)^{n-k+1} c_k^{(n-k+1)} \phi'^{(1)} + \\ & +(-1)^k c_1^{(0)} c_{k-1}^{(n-k+2)} + c_{k-1}^{(n-k+2)} c'^{(0)}_1 \Big)\lambda \,. \end{aligned} \tag{7.46}$$

Then nilpotence of the BRST transformation of the $(k-1)^{th}$ generation ghost implies:

$$\delta_{k-1} c_k{}^{(n-k+1)} = \Big((-1)^{k+1} c_1^{(0)} c_k{}^{(n-k+1)} + c_k{}^{(n-k+1)} c'^{(0)}_1 \Big)\lambda \,. \tag{7.47}$$

We finally note that the BRST transformations (7.47) have the following on-shell zero modes:

$$\delta_k c_k^{(n-k+1)} = \Big(dc_{k+1}^{(n-k)} + \phi^{(1)} c_{k+1}^{(n-k)} + (-1)^{n-k+2} c_{k+1}^{(n-k)} \phi'^{(1)} \Big)\lambda \,. \tag{7.48}$$

We recover the full BRST transformation (7.45) as the sum of (7.47) and (7.48), $\delta^{(k)} c_k^{(n-k+1)} = \sum_{j=1}^{k} \delta_j c_k^{(n-k+1)} = (\delta_{k-1} + \delta_k) c_k^{(n-k+1)}$, where we used the fact that all transformations with the exception those of levels $k-1$ and k act trivially on $c_k^{(n-k+1)}$.

Note that the Grassmann parity of the ghosts depends on the generation number

$$g(c_k^{(n-k+1)}) = (-)^k \,. \tag{7.49}$$

We now set to zero all forms of ranks higher than 3, which eliminates all ghosts except for those of ranks between zero and three. The ansatz:

$$\hat{\phi} = \begin{pmatrix} c_1^{(0)} + \phi^{(1)} - \phi^{*(2)} + c_1^{*(3)} & c_{n+1}^{(0)} + c_n^{(1)} + c_{n-1}^{(2)} + c_{n-2}^{(3)} \\ (-)^{n+1} c_{n-2}^{*(0)} + c_{n-1}^{*(1)} - (-)^n c_n^{*(2)} + c_{n+1}^{*(3)} & c_1'^{(0)} + \phi'^{(1)} - (-)^n \phi'^{*(2)} + (-)^n c_1'^{*(3)} \end{pmatrix} \tag{7.50}$$

shows that the resulting BV action coincides with the extended action (3.34). It is also easy to check that (7.50) define Darboux coordinates for the odd symplectic form (4.66). Moreover, because of equation (7.49), the degree of the extended string field is 1.

7.5 On the equivalence of the extended actions with $n \geq 3$

Given the fact that the classical actions and physical fields of the theories with relative grading $n \geq 3$ are identical, we could attempt to relate the extended actions by a canonical transformation. As shown in the previous subsection, in these cases the extended action can be obtained by adding a vanishing $(n+1)$-form in the sector $\mathrm{Hom}(E_a, E_b)$. Without this trick, one would conclude that the BV action is the graded sum of the two master actions corresponding to E_a and E_b. Therefore, it seems natural to inquire whether the extended action can be expressed in this form. We will first isolate from the extended action the graded sum part and then search for a canonical transformation which annihilates the remnant.

Let the extended field $\hat{\phi}$ for some grade difference $n \geq 3$ be

$$\hat{\phi} = \hat{\phi}_0 + \hat{c} + \hat{c}^* \,, \tag{7.51}$$

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where:

$$
\begin{aligned}
\hat{\phi}_0 &= \begin{pmatrix} c_1^{(0)} + \phi^{(1)} + \phi^{*(2)} + c_1^{*(3)} & 0 \\ 0 & c'^{(0)}_1 + \phi'^{(1)} + \phi'^{*(2)} + c'^{*(3)}_1 \end{pmatrix} \equiv \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix} \\
\hat{c} &= \begin{pmatrix} 0 & c_{k+3}^{(0)} + c_{k+2}^{(1)} + c_{k+1}^{(2)} + c_k^{(3)} \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \\
\hat{c}^* &= \begin{pmatrix} 0 & 0 \\ c_{k+3}^{*(0)} + c_{k+2}^{*(1)} + c_{k+1}^{*(2)} + c_k^{*(3)} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ c^* & 0 \end{pmatrix} .
\end{aligned}
\tag{7.52}
$$

Then the extended action is:

$$
S_e = S_e(\hat{\phi}_0) + \int_L \mathrm{str}_e \left[\hat{c}^* * (d\hat{c} + \hat{\phi}_0 * \hat{c} + \hat{c} * \hat{\phi}_0) \right], \tag{7.53}
$$

where $S_e(\hat{\phi}_0)$ is the graded sum of the two BV actions corresponding to E_a and E_b. We now look for a canonical transformation which annihilates the last term of the previous equation:

$$
\begin{aligned}
\Delta S_e \equiv S_e(\hat{\phi}) - S_e(\hat{\phi}_0) &= (-1)^n \int_L \mathrm{tr}_b \left[c^* * \left(dc + \phi * c + c * \phi' \right) \right] \\
&= \int_L \mathrm{tr}_a \left[c * \left(-dc^* + \phi' * c^* + c^* * \phi \right) \right].
\end{aligned}
\tag{7.54}
$$

Let Ψ be the generator of this canonical transformation, a Grassmann-odd function of fields and antifields whose ghost number equals -1. If $\mathrm{ad}_\Psi(F) := \{\Psi, F\}$ is the left adjoint action of Ψ on Grassmann-valued functions F of fields and antifields, then the associated one-parameter group of canonical transformations acts as $e^{t\,\mathrm{ad}_\Psi} F$. Notice that the BV bracket of two monomials of degrees p and q in fields and antifields gives either zero or a homogeneous polynomial of degree $p+q-2$. Since (7.54) contains terms of degrees 2 and 3 in fields and antifields, it follows that the adjoint action of a term of order p in the Taylor expansion of Ψ will produce zero or a sum of polynomials of degrees p and $p+1$. This implies that the only source of quadratic terms in $e^{t\,\mathrm{ad}_\Psi} \Delta S_e$ is the quadratic term in the expansion of Ψ acting an arbitrary number of times on the quadratic term in the expansion of ΔS_e. Therefore, a crucial property of the desired generator Ψ is that the action of $e^{t\,\mathrm{ad}_\Psi}$ on the quadratic part of ΔS_e vanishes.

If we also also require that $S_e(\hat{\phi}_0)$ be invariant under the canonical transformation we are looking for, it follows that the only choice (up to a factor) for the quadratic term in Ψ is

$$
\Psi_2 = \int_L \mathrm{str}_e[\hat{c} * \hat{c}^*] = \int_L \mathrm{tr}_a[c * c^*] = -(-)^n \int_L \mathrm{tr}_b[c^* * c], \tag{7.55}
$$

since $S_e(\hat{\phi}_0)$ is independent of the pair $(\hat{c}, \hat{c}^*)$. The fact that ΔS_e is linear in fields and antifields,

$$
\mathrm{tr}_a[c \frac{\partial}{\partial c} \Delta S_e] = \Delta S_e \qquad \mathrm{tr}_b[c^* \frac{\partial}{\partial c^*} \Delta S_E] = \Delta S_e \tag{7.56}
$$

implies that:

$$
\mathrm{ad}_{\Psi_2} \Delta S_e \equiv \{\Psi_2,\, \Delta S_e\} = -(1 + (-1)^n) \Delta S_e \,. \tag{7.57}
$$

It follows that a finite transformation of ΔS_e has the form:

$$e^{t\,\mathrm{ad}_{\Psi_2}}\Delta S_e = \frac{1}{2}\left(1-(-1)^n\right)\Delta S_e + \frac{1}{2}\left(1+(-1)^n\right)e^{-2t}\Delta S_e\,. \tag{7.58}$$

We therefore find that for odd n the canonical transformation generated by Ψ_2 maps ΔS_e to itself. In particular, since Ψ_2 was unique up to a factor, this implies that ΔS_e cannot be removed. Hence S_e cannot be canonically transformed into a difference of CS actions for $n = 2k+1 \geq 3$.

For even n, the transformation (7.58) multiplies ΔS_e by the factor e^{-2t} which is non-vanishing for any finite t. In the limit $t \to +\infty$, ΔS_e is mapped to zero. Thus, by choosing $\Psi = \Psi_2$ it seems that we can remove the full ΔS_e. Unfortunately, the limit $t \to +\infty$ describes a point in the closure of the space of canonical transformations and it is not clear whether such a transformation is indeed allowed.

Thus, the extended actions with relative grading $n \geq 3$ (with n of a given parity) do not seem to be canonically equivalent, even though their classical counterparts are. However, this does not necessarily mean that the physical phenomena described by them are different, since the differences between the actions arise, as shown in the previous section, by gauging the symmetries associated to a vanishing field. In [48] a rather similar situation was analyzed (the classical action was taken to vanish and the BV action was constructed using the gauge symmetries of a 5-form field strength coupled to gravity, in four dimensions). The conclusion of the analysis was that the vanishing of physical observables survives quantization and the fields introduced by the BV construction contribute only to gauge dependent correlation functions. The analogy with the situation in that paper suggests that once quantization is performed, the off-diagonal sector $\mathrm{Hom}(E_a, E_b)$ does not contribute to any observable, and the equivalence of classical actions with $n \geq 3$ extends to an equivalence between the algebras of observables at the quantum level.

8. Relation between the constructive and geometric approach

As discussed in sections 4 and 7, the role of S_e as a tree-level BV action for graded D-brane pairs can be understood from two quite different perspectives, a 'top-down' approach which makes use of the geometric formalism and a 'bottom-up' approach which uses the standard homological approach. While the geometric formalism has the advantage of extreme generality (for example, it applies to an arbitrary number of graded D-branes, for which direct computation in components is impractical) and conceptual elegance, the discussion of section 7 is more explicit and constructive (providing, in particular, a deeper understanding of the origin of various ghosts). In this section we explain the relation between the two descriptions, and give a more synthetic formulation of the recursive computations of the previous section.

It is clear from the direct analysis of the previous section that the gauge algebra of our models is generally reducible. Therefore, the BRST procedure extends S by adding the first order action S_1, which is built recursively from a tower of ghosts of various generations, accompanied by the corresponding antifields. This leads to successive extension of the

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string field ϕ to enlarged collections of fields $\boldsymbol{\phi}^{(k)}$ and antifields $\boldsymbol{\phi}^{*(k)}$. At each step k, one builds the k^{th} approximation $S_1^{(k)}$ (defined on $\mathcal{L}^{(k)} \oplus \mathcal{N}^{(k)}$) to the first order action:

$$S_1^{(k)}(\boldsymbol{\phi}^{(k)}, \boldsymbol{\phi}^{*(k)}) = -\omega(\boldsymbol{\phi}^{*(k)}, \mathbf{q}^{(k)}(\boldsymbol{\phi}^{(k)}))\,, \tag{8.1}$$

where $\mathbf{q}^{(k)}$ is the BRST generator at order k (identified with a nonlinear operator acting on the linear space of truncated fields). The procedure stops at the step $k_m = \sigma + 1$ dictated by order of reducibility σ of the gauge algebra. Then $\mathbf{q}^{(k_m)} = \mathbf{q}$ and $S_1 = S_1^{(k_m)} = -\omega(\boldsymbol{\phi}^*, \mathbf{q}(\boldsymbol{\phi}))$. In view of (4.84), the full master action $S_{BV} = S_e$ is obtained from $S + S_1$ by adding the following term quadratic in in antifields:

$$S_2(\boldsymbol{\phi}^*, \boldsymbol{\phi}) = \omega(\boldsymbol{\phi}, \boldsymbol{\phi}^* * \boldsymbol{\phi}^*)\,. \tag{8.2}$$

To describe this in the geometric language of section 4, we consider the expansion $\boldsymbol{\phi} = \sum_{s\geq 0} \phi_s$, with $\phi_s \in M_s$. The BRST operator (4.83) has the form:

$$\mathbf{q}(\boldsymbol{\phi}) = \oplus_{s\geq 0}\mathbf{q}_s(\boldsymbol{\phi})\,, \tag{8.3}$$

with $\mathbf{q}_s(\boldsymbol{\phi}) \in \Pi M_s = P_s$. Expanding (4.83) gives:

$$\mathbf{q}_s(\boldsymbol{\phi}) = -d\phi_{s+1} - \sum_{i+j=s+1} \phi_i * \phi_j\,. \tag{8.4}$$

Consider the subspaces:

$$\mathcal{L}^{(k)} := \oplus_{0\leq s\leq k} M_s\,, \qquad \mathcal{R}^{(k)} := \oplus_{s>k} M_s\,, \tag{8.5}$$

which give complementary ascending and descending sequences of approximations for $\mathcal{L}$:

$$\begin{aligned}
&\mathcal{L}^{(0)} \subset \mathcal{L}^{(1)} \subset \mathcal{L}^{(2)} \subset \cdots \subset \mathcal{L}^{(k_m)} = \mathcal{L}\\
&0 = \mathcal{R}^{(k_m)} \subset \mathcal{R}^{(k_m-1)} \subset \cdots \subset \mathcal{R}^{(1)} \subset \mathcal{R}^{(0)}\,,\\
&\mathcal{L}^{(k)} \oplus \mathcal{R}^{(k)} = \mathcal{L}\,.
\end{aligned} \tag{8.6}$$

We also define $\mathcal{N}^{(k)} = \oplus_{-k\leq s<0} M_s$, which give an ascending sequence of approximations for $\mathcal{N}$. Then the k-th approximation to $\mathbf{q}$ is obtained as:

$$\mathbf{q}^{(k)} = P^{(k)}\mathbf{q}|_{\mathcal{L}^{(k)}}\,, \tag{8.7}$$

where $P^{(k)}$ is the projector of $\Pi\mathcal{L}$ onto $\Pi\mathcal{L}^{(k)}$, parallel with the subspace $\Pi\mathcal{R}^{(k)}$. Defining:

$$\boldsymbol{\phi}^{(k)} := \sum_{0\leq s\leq k} \phi_s \in \mathcal{L}^{(k)}\,, \qquad \boldsymbol{\phi}^{*(k)} := \sum_{0\leq s\leq k} \phi_s^* \in \mathcal{N}^{(k)}\,, \tag{8.8}$$

we can write:

$$\mathbf{q}^{(k)}(\boldsymbol{\phi}^{(k)}) = \oplus_{0\leq s\leq k}\mathbf{q}_s^{(k)}(\boldsymbol{\phi}^{(k)})\,, \tag{8.9}$$

with:

$$\mathbf{q}_s^{(k)}(\boldsymbol{\phi}^{(k)}) = -d\phi_{s+1} - \sum_{\substack{i+j=s+1\\ 0\leq i,j\leq k}} \phi_i * \phi_j\,. \tag{8.10}$$

This explicit expression for $\mathbf{q}^{(k)}$ allows us to recover the truncation $S_1^{(k)}$ of the first order action upon applying the prescription (8.1). For comparison with the case of D-brane pairs, we list the first three approximations $k = 1, 2, 3$.

First approximation. The first order extended field and antifield are:

$$\boldsymbol{\phi}^{(1)} = \phi + c_1 \,, \qquad \boldsymbol{\phi}^{*(1)} = \phi^* + c_1^* \,. \tag{8.11}$$

Expanding (8.10) gives:[28]

$$\begin{aligned} -\mathbf{q}_0^{(1)}(\boldsymbol{\phi}^{(1)}) &= dc_1 + \phi * c_1 + c_1 * \phi \\ -\mathbf{q}_1^{(1)}(\boldsymbol{\phi}^{(1)}) &= c_1 * c_1 \,. \end{aligned} \tag{8.12}$$

For D-brane pairs with relative grading $n = 0$ the gauge algebra is irreducible ($o = 0$) and $S_1 = S_1^{(1)}$.

Second approximation. For this, one introduces second generation ghosts c_2 and antighost c_2^*. The BRST transformations of the extended field $\boldsymbol{\phi}^{(2)} = \phi + c_1 + c_2$ have the expanded form:

$$\begin{aligned} -\mathbf{q}_0^{(2)}(\boldsymbol{\phi}^{(2)}) &= dc_1 + \phi * c_1 + c_1 * \phi \\ -\mathbf{q}_1^{(2)}(\boldsymbol{\phi}^{(2)}) &= dc_2 + \phi * c_2 + c_2 * \phi + c_1 * c_1 \\ -\mathbf{q}_2^{(2)}(\boldsymbol{\phi}^{(2)}) &= c_2 * c_1 + c_1 * c_2 \,. \end{aligned} \tag{8.13}$$

The second approximation $S_1^{(2)}$ to S_1 is given by (8.1) with the antifields $\boldsymbol{\phi}^{*(2)} = \phi^* + c_1^* + c_2^*$. For D-brane pairs of relative grading $n = 1$, the gauge algebra is first stage reducible ($o = 1$) and $S_1 = S_1^{(2)}$.

Third approximation. We continue by adding a third generation ghost c_3 and antifield c_3^*. This gives the extended field $\boldsymbol{\phi}^{(3)} = \phi + c_1 + c_2 + c_3$, whose components has the BRST transformations (8.10):

$$\begin{aligned} -\mathbf{q}_0^{(3)}(\boldsymbol{\phi}^{(3)}) &= dc_1 + \phi * c_1 + c_1 * \phi \\ -\mathbf{q}_1^{(3)}(\boldsymbol{\phi}^{(3)}) &= dc_2 + \phi * c_2 + c_2 * \phi + c_1 * c_1 \\ -\mathbf{q}_2^{(3)}(\boldsymbol{\phi}^{(3)}) &= dc_3 + \phi * c_3 + c_3 * \phi + c_2 * c_1 + c_1 * c_2 \\ -\mathbf{q}_3^{(3)}(\boldsymbol{\phi}^{(3)}) &= c_2 * c_2 + c_1 * c_3 + c_3 * c_1 \,. \end{aligned} \tag{8.14}$$

The third approximation $S_1^{(3)}$ is constructed from (8.1) with the antifields $\boldsymbol{\phi}^{*(3)} = \phi^* + c_1^* + c_2^* + c_3^*$. This coincides with the full first order action for the case of relative grading $n = 3$. For D-brane pairs with relative grading $n = 2$, the gauge algebra is second order reducible ($o = 2$) and $S_1 = S_1^{(3)}$.

[28] In the previous section we used the infinitesimal BRST transformation denoted by δ. As usual, the infinitesimal transformations are given by the action of the generator multiplied by the parameter of the transformation. In our case, at the first step in the iterative procedure, we have:

$$\delta\boldsymbol{\phi}^{(1)} \equiv -\mathbf{q}^{(1)}(\boldsymbol{\phi}^{(1)})\lambda \,,$$

where λ is a Grassmann-odd constant which plays the role of parameter for the BRST transformations. Projecting this relation onto $M_0(\hat{0})$ and $M_1(\hat{1})$ we find the infinitesimal transformations used previously:

$$\delta^{(1)}\phi = (dc_1 + \phi * c_1 + c_1 * \phi)\lambda \,, \qquad \delta^{(1)}c_1 = (c_1 * c_1)\lambda \,.$$

To check agreement with previous computations, let us first consider the case $n = 1$. It is not hard to check that upon inserting (7.15) into (8.13) one recovers equation (7.10) of the previous section. For $n = 2$ one recovers the BRST transformations (7.27) upon inserting (7.30) into (8.14). It is also easy to check that the difference between $\delta^{(k)} = -\mathbf{q}^{(k)}\lambda$ and $\delta^{(k-1)} = -\mathbf{q}^{(k-1)}\lambda$ of this section produces δ_k of the previous section. It is clear that the covariant formulation given above can be generalized to systems containing more than two graded D-branes.

9. Conclusions and directions for further research

We studied graded D-brane systems along the lines proposed in [6], showing that the extended action written down in that paper plays the role of classical master action for the string field theory of such backgrounds. We gave a completely general proof of the master equation by making use of a certain $\mathbb{Z}$-graded version of the geometric BV formalism, which is based on the concept of graded supermanifolds recently introduced by T. Voronov. We argued that graded supermanifolds are the correct framework for a covariant description of BV systems, and discussed the basics of the geometric approach within this context. These results are of independent interest for foundational studies of BV quantization.

We also performed a direct construction of the master action for the case of graded D-brane pairs. This allowed us to identify the various components of the extended string field as ghosts and antifields produced by the BV procedure, and explain their origin in the reducibility of the gauge algebra. Upon using the formalism of [2] and [3], we analyzed formation of D-brane composites in systems of two graded D-branes, and in particular gave a rigorous construction of acyclic condensates for the case of unit relative grading. This gives a detailed implementation and generalization of ideas proposed in [14], though in a somewhat different context.

For the case of graded D-brane pairs, we showed that the six classical theories corresponding to various relative grades arise through different choices of ghost grading and classical gauge for two underlying master actions, the extended Chern-Simons and extended super-Chern-Simons functionals. We also showed that these theories are inequivalent in the case when the underlying special lagrangian three-cycle is topologically nontrivial. This sheds new light on the 'mod 6 periodicity' of the D-brane grade discussed from a worldsheet perspective in [1].

Our results can be viewed as a starting point for the quantization of such systems. As already pointed out in [6], the string field theory of graded D-branes gives a concrete representation of points of the *extended* boundary moduli space, thereby holding the promise for a better understanding of its physical significance. This should be of direct relevance for the homological mirror symmetry programme. A quantum analysis of our theories around such backgrounds should lead to new physical information, as well as to certain extensions of various geometric invariants. This and related matters are the subject of ongoing research. Here we only note that a thorough study away from the large radius limit must take into account destabilization and other instanton effects [9, 10, 49].

Let us finally mention that a similar analysis can be carried out for the open B-model, in which case one deals with (graded) D6-branes wrapping the entire Calabi-Yau manifold. In that case, the relevant string field theory is a graded version of holomorphic Chern-Simons theory [3, 5]. It is clear that all of our constructions have a parallel in such situations, provided that one makes the appropriate modifications. Since the B model does not receive instanton corrections, the BV systems associated to graded B-type branes should be viewed as a description of the associated deformation theory; for example, they allow for a classification of the local string field observables, which can be used to deform such models. Some of these issues are currently under investigation.

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A. Graded supermodules and super-bimodules

Consider an associative superalgebra G ($\mathbb{Z}_2$-graded associative algebra with a unit). We remind the reader that a *right/left supermodule* U over G is a $\mathbb{Z}_2$-graded right/left G-module such that the scalar multiplication $U \times G \to U$ (respectively $G \times U \to U$) is homogeneous of degree zero:

$$\deg(u\alpha) = \deg u + \deg \alpha \quad \text{respectively} \;\; \deg(\alpha u) = \deg u + \deg \alpha \tag{A.1}$$

where deg denotes the grading on U or G. A *super-bimodule* U is simultaneously a left and right G-module such that the left and right scalar multiplications are compatible:

$$\alpha u = (-1)^{\deg \alpha \ \deg u} u\alpha \,. \tag{A.2}$$

It is clear that this relation determines one scalar multiplication given the other, so a left or right supermodule can be made into a super-bimodule in a unique manner.

B. Left and right vector fields

Consider a De Witt-Rogers supermanifold M modeled over the algebra of constants G. The space $\mathcal{F}(M, G)$ of G-valued (smooth) functions is then a super-bimodule over G with respect to pointwise scalar multiplication:

$$(F\alpha)(p) = F(p)\alpha \,, \qquad (\alpha F)(p) = \alpha F(p) \,. \tag{B.1}$$

It is also a ring with respect to pointwise multiplication of functions:

$$(FG)(p) = F(p)G(p) \,. \tag{B.2}$$

Combining the two structures, we obtain a superalgebra over G; this statement means that one has relations such as $\alpha F G = (-1)^{\epsilon_\alpha \epsilon_F} F \alpha G = (-1)^{\epsilon_\alpha(\epsilon_F+\epsilon_G)} F G \alpha$.

A left (right) vector field on M is a left (right) graded derivation of this algebra. For left vector fields, this means:

$$X_l(F\alpha) = X_l(F)\alpha\,, \qquad X_l(FG) = X_l(F)G + (-1)^{\epsilon_F \epsilon_{X_l}} F X_l(G)\,, \tag{B.3}$$

while for right vector fields one requires:

$$X_r(\alpha F) = \alpha X_r(F)\,, \qquad X_r(FG) = F X_r(G) + (-1)^{\epsilon_G \epsilon_{X_r}} X_r(F)G\,. \tag{B.4}$$

In these relations, ϵ_{X_l} and ϵ_{X_r} are the vector field parities. One also defines scalar multiplications:

$$(\alpha X_l)(F) := \alpha X_l(F)\,, \qquad (Y_l \alpha)(F) = Y_l(F)\alpha\,. \tag{B.5}$$

With these operations and grading (and the obvious definition of addition), the space $\mathcal{X}_l(M)$ of left derivations of $\mathcal{F}(M)$ is a left G-supermodule, while the space $\mathcal{X}_r(M)$ of right derivations is a right G-supermodule.

It is common procedure to identify left and right vector fields on M in the following manner. If X_l is a left vector field, one defines $\phi(X_l)$ by:

$$\phi(X_l)(F) := (-1)^{\epsilon_X \epsilon_F} X_l(F)\,. \tag{B.6}$$

It is the easy to check that ϕ is a degree zero isomorphism between the graded abelian groups $(X_l(M), +)$ and $(X_r(M), +)$. Moreover, one has the property:

$$\phi(\alpha X_l) = (-1)^{\epsilon_\alpha \epsilon_X} \phi(X_l)\alpha\,. \tag{B.7}$$

This allows one to identify $\mathcal{X}_l(M)$ and $\mathcal{X}_r(M)$ to a single abstract space $\mathcal{X}(M)$, the space of vector fields on M. This space inherits left and right module structures from $X_l(M)$ and $X_r(M)$, and relation (B.7) shows that the two structures are compatible. Thus $\mathcal{X}(M)$ is a super-bimodule over G. An element X of $\mathcal{X}(M)$ can be viewed as a pair (X_l, X_r) with $X_l \in \mathcal{X}_l(G)$ and $X_r \in \mathcal{X}_r(G)$, where X_l and X_r are ϕ-related, i.e. $X_r = \phi(X_l)$. In this situation, one uses the notation:

$$X_l := \overrightarrow{X}\,, \qquad X_r := \overleftarrow{X}\,, \tag{B.8}$$

and writes $\overrightarrow{X}\, F$ for $X_l(F)$ and $F\, \overleftarrow{X}$ for $X_r(F)$. With these notations, the ϕ-relatedness condition becomes:

$$\overrightarrow{X}\, F = (-1)^{\epsilon_X \epsilon_F} F\, \overleftarrow{X}\,. \tag{B.9}$$

This is the rigorous construction behind the conventions used in section 4.1. We warn the reader that the vector fields ∂_a^l and ∂_a^r defined in that section do *not* form a ϕ-related pair, in spite of the similarity with the notation used in this appendix. In the body of this paper, we use exclusively the convention that left and right vector fields are identified to abstract objects X, and the left and right components of X are denoted by superscript arrows as explained above.

C. Direct check of the master equation for D-brane pairs

In this appendix we exemplify the steps needed for the proof that the BV action satisfies the master equation. We choose to take here an easier (but equivalent) path, namely we prove that the BRST transformations are nilpotent. More precisely, we show that:

$$\delta(\lambda_2)\delta(\lambda_1)\hat{\phi} = \{\{\hat{\phi}, S_{BV}\lambda_1\}, S_{BV}\lambda_2\} = 0\,, \tag{C.1}$$

where λ_1, λ_2 are anti-commuting constants.[29] For the case with relative grading $n = 1$, consider the BRST transformations of $\phi^{(1)}$:

$$\begin{aligned}
\delta(\lambda_2)\delta(\lambda_1)\phi^{(1)} &= \Big\{\Big(dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)}c_1^{(0)} - c_1^{(1)}\phi^{(0)} - c_2^{(0)}\phi^{*(1)}\Big)\,\lambda_1, S_{BV}\lambda_2\Big\}\\
&= \Big[-d((c_1^{(0)})^2 + c_2^{(0)}\phi^{(0)}) + ((c_1^{(0)})^2 + c_2^{(0)}\phi^{(0)})\phi^{(1)} - \phi^{(1)}((c_1^{(0)})^2 + c_2^{(0)}\phi^{(0)}) +\\
&\quad + (dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)}c_1^{(0)} - c_1^{(1)}\phi^{(0)} - c_2^{(0)}\phi^{*(1)})c_1^{(0)} +\\
&\quad + c_1^{(0)}(dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)}c_1^{(0)} - c_1^{(1)}\phi^{(0)} - c_2^{(0)}\phi^{*(1)}) +\\
&\quad + (c_1^{(0)}c_1^{(1)} + c_1^{(1)}c_1^{\prime(0)} + dc_2^{(0)} + \phi^{(1)}c_2^{(0)} - c_2^{(0)}\phi^{\prime(1)})\phi^{(0)} +\\
&\quad + (c_1^{(0)}c_2^{(0)} - c_2^{(0)}c_1^{\prime(0)})\phi^{*(1)} + c_1^{(1)}(\phi^{(0)}c_1^{(0)} - c_1^{\prime(0)}\phi^{(0)}) -\\
&\quad - c_2^{(0)}(-d\phi^{(0)} + \phi^{(0)}\phi^{(1)} - \phi^{\prime(1)}\phi^{(0)} - \phi^{*(1)}c_1^{(0)} - c_1^{\prime(0)}\phi^{*(1)})\Big]\lambda_1\lambda_2 = 0\,.
\end{aligned} \tag{C.2}$$

We similarly have:

$$\begin{aligned}
\delta(\lambda_2)\delta(\lambda_1)\phi^{(0)} &= \Big\{\Big(\phi^{(0)}c_1^{(0)} + c_1^{\prime(0)}\phi^{(0)}\Big)\,\lambda_1, S_{BV}\lambda_2\Big\}\\
&= \Big[-(\phi^{(0)}c_1^{(0)} - c_1^{\prime(0)}\phi^{(0)})c_1^{(0)} + \phi^{(0)}(c_2^{(0)}\phi^{(0)} + (c_1^{(0)})^2) -\\
&\qquad - ((c_1^{\prime(0)})^2 + \phi^{(0)}c_2^{(0)})\phi^{(0)} - c_1^{\prime(0)}(\phi^{(0)}c_1^{(0)} - c_1^{\prime(0)}\phi^{(0)})\Big]\lambda_1\lambda_2 = 0\,,
\end{aligned} \tag{C.3}$$

as well as:

$$\begin{aligned}
\delta(\lambda_2)\delta(\lambda_1)\phi^{(2)} =& \Big\{\Big(dc_1^{(1)} + \phi^{(2)}c_1^{\prime(0)} - c_1^{(0)}\phi^{(2)} + \phi^{(1)}c_1^{(1)} +\\
&\qquad + c_1^{(1)}\phi^{\prime(1)} + \phi^{*(2)}c_2^{(0)} + c_2^{(0)}\phi^{\prime *(2)}\Big)\lambda_1, S_{BV}\lambda_2\Big\}\\
=& \Big[-d(c_1^{(0)}c_1^{(1)} + c_1^{(1)}c_1^{\prime(0)} + dc_2^{(0)} + \phi^{(1)}c_2^{(0)} - c_2^{(0)}\phi^{\prime(1)}) -\\
&\quad - \phi^{(1)}(c_1^{(0)}c_1^{(1)} + c_1^{(1)}c_1^{\prime(0)} + dc_2^{(0)} + \phi^{(1)}c_2^{(0)} - c_2^{(0)}\phi^{\prime(1)}) -\\
&\quad - (c_1^{(0)}c_1^{(1)} + c_1^{(1)}c_1^{\prime(0)} + dc_2^{(0)} + \phi^{(1)}c_2^{(0)} - c_2^{(0)}\phi^{\prime(1)})\phi^{\prime(1)} +\\
&\quad + (dc_1^{(1)} + \phi^{(2)}c_1^{\prime(0)} - c_1^{(0)} + c_1^{(1)}\phi^{\prime(1)} + \phi^{(1)}c_1^{(1)} + \phi^{*(2)}c_2^{(0)} + c_2^{(0)}\phi^{\prime *(2)})c_1^{\prime(0)} +\\
&\quad + c_1^{(0)}(dc_1^{(1)} + \phi^{(2)}c_1^{\prime(0)} - c_1^{(0)} + c_1^{(1)}\phi^{\prime(1)} + \phi^{(1)}c_1^{(1)} + \phi^{*(2)}c_2^{(0)} + c_2^{(0)}\phi^{\prime *(2)}) +\\
&\quad + ((c_1^{(0)})^2 + c_2^{(0)}\phi^{(0)})\phi^{(2)} - \phi^{(2)}((c_1^{\prime(0)})^2 + \phi^{(0)}c_2^{(0)}) -
\end{aligned}$$

[29] Each block component in the matrix $\hat{\phi}$ can be viewed locally as a matrix of forms. For example, $(\phi^{(1)})^{ji}$ and its antifield $(\phi^{*(2)})^{ij}$ have indices $i, j = 1, \ldots rkE_a$ while $(\phi^{(0)})^{ik}$ and its antifield $(\phi^{(3)})^{ki}$ have indices $i = 1, \ldots rkE_a$, $k = 1, \ldots rkE_b$. To avoid complicated notation we suppress all such indices.

$$
\begin{aligned}
&- c_1^{(1)}(dc_1^{'(0)} + \phi'^{(1)} c_1^{'(0)} - c_1^{'(0)}\phi'^{(1)} - \phi^{(0)} c_1^{(1)} - \phi^{*(1)} c_2^{(0)}) + \\
&+ (dc_1^{(0)} + \phi^{(1)} c_1^{(0)} - c_1^{(0)}\phi^{(1)} + c_1^{(1)}\phi^{(0)} - c_2^{(0)}\phi^{*(1)})c_1^{(1)} + \\
&+ \phi^{*(2)}(c_1^{(0)} c_2^{(0)} - c_2^{(0)} c_1^{'(0)}) - (c_1^{(0)} c_2^{(0)} - c_2^{(0)} c_1^{'(0)})\phi'^{*(2)} + \\
&+ (d\phi^{(1)} + \phi^{(1)2} + \phi^{(2)}\phi^{(0)} - \phi^{*(2)} c_1^{(0)} - c_1^{(0)}\phi^{*(2)} - c_1^{(1)}\phi^{*(1)} + c_2^{(0)} c_1^{*(2)})c_2^{(0)} + \\
&+ c_2^{(0)}(-d\phi'^{(1)} - \phi'^{(1)2} - \phi^{(0)}\phi^{(2)} - c_1^{'(0)}\phi'^{*(2)} - \phi'^{*(2)} c_1^{'(0)} + \phi^{*(1)} c_1^{(1)} - \\
&\qquad - c_1^{*(2)} c_2^{(0)})\Big]\lambda_1\lambda_2 = 0\,. \qquad \text{(C.4)}
\end{aligned}
$$

Nilpotence of the BRST transformations in the ghost sector follows from similar calculations. For instance:

$$
\begin{aligned}
\delta_{\lambda_2}\delta_{\lambda_1} c_2^{(0)} &= \Big\{(c_1^{(0)} c_2^{(0)} - c_2^{(0)} c_1^{'(0)})\lambda_1, S_{BV}\lambda_2\Big\} \\
&= \Big[c_1^{(0)}(c_1^{(0)} c_2^{(0)} - c_2^{(0)} c_1^{'(0)}) - ((c_1^{(0)})^2 + c_2^{(0)}\phi^{(0)})c_2^{(0)} + \\
&\quad + c_2^{(0)}((c_1^{'(0)})^2 + \phi^{(0)} c_2^{(0)}) + ((c_1^{(0)} c_2^{(0)} - c_2^{(0)} c_1^{'(0)})c_1^{'(0)})\Big]\lambda_1\lambda_2 = 0\,. \qquad \text{(C.5)}
\end{aligned}
$$

To prove the master equation, one must also analyze the transformation of antifields.

Let us now exemplify nilpotence of the bracket for the case of relative grading $n = 2$. For $\phi^{(1)}$, one has:

$$
\begin{aligned}
\delta_{\lambda_2}\delta_{\lambda_1}\phi^{(1)} &= \Big\{\Big(dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)} c_1^{(0)} - c_2^{(0)} c_1^{*(1)} - c_2^{(1)}\phi^{*(0)}\Big)\lambda_1, S_{BV}\lambda_2\Big\} \\
&= \Big[-d((c_1^{(0)})^2 - c_2^{(0)}\phi^{*(0)}) - \phi^{(1)}((c_1^{(0)})^2 - c_2^{(0)}\phi^{*(0)}) + \\
&\quad + ((c_1^{(0)})^2 - c_2^{(0)}\phi^{*(0)})\phi^{(1)} + \\
&\quad + c_1^{(0)}(dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)} c_1^{(0)} - c_2^{(0)} c_1^{*(1)} - c_2^{(1)}\phi^{*(0)}) + \\
&\quad + (dc_1^{(0)} - c_1^{(0)}\phi^{(1)} + \phi^{(1)} c_1^{(0)} - c_2^{(0)} c_1^{*(1)} - c_2^{(1)}\phi^{*(0)})c_1^{(0)} + \\
&\quad + (c_1^{(0)} c_2^{(0)} + c_2^{(0)} c_1^{'(0)})c_1^{*(1)} - \\
&\quad - c_2^{(0)}(d\phi^{*(0)} + \phi'^{(1)}\phi^{*(0)} - \phi^{*(0)}\phi^{(1)} - c_1^{*(1)} c_1^{(0)} + c_1^{'(0)} c_1^{*(1)}) - \\
&\quad - (dc_2^{(0)} - c_2^{(0)}\phi'^{(1)} + \phi^{(1)} c_2^{(0)} - c_1^{(0)} c_2^{(1)} + c_2^{(1)} c_1^{'(0)})\phi^{*(0)} + \\
&\quad + c_2^{(1)}(\phi^{*(0)} c_1^{(0)} + c_1^{'(0)}\phi^{*(0)})\Big]\lambda_1\lambda_2 = 0\,. \qquad \text{(C.6)}
\end{aligned}
$$

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